

HCMU metrics with cusp singularities and conical singularities

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Abstract

An HCMU metric is a conformal metric which has a finite number of singularities on a compact Riemann surface and satisfies the equation of the extremal Kähler metric. In this paper, we give a necessary and sufficient condition for the existence of a kind of HCMU metrics which has both cusp singularities and conical singularities.

1 Introduction

The extremal Kähler metric on a Kähler manifold was defined in [1] by Calabi. The aim is to find the “best” metric in a fixed Kähler class on a compact Kähler manifold M . In a fixed Kähler class, an extremal Kähler metric is the critical point of the following Calabi energy functional

$$\mathcal{C}(g) = \int_M R^2 dg,$$

where R is the scalar curvature of the metric g in the Kähler class. The Euler-Lagrange equation of $\mathcal{C}(g)$ is $R_{,\alpha\beta} = 0$, where $R_{,\alpha\beta}$ is the second-order $(0, 2)$ covariant derivative of R . When M is a compact Riemann surface without boundary, Calabi proved that an extremal Kähler metric is a CSC(constant scalar curvature) metric in [1]. This coincides with the classical uniformization theorem, which says that there exists a CSC metric in each fixed Kähler class of Riemann surface without boundary.

On the other hand, there have been many attempts to generalize the classical uniformization theorem to surfaces with boundaries. The main focus, started by the independent work of Troyanov[11] and McOwen[9], has been to study the existence or nonexistence of constant curvature metrics on surfaces with conical singularities. But in general one should not expect to get a clear-cut statement about the existence(or nonexistence) of solutions, since the constant curvature equation is overdetermined in this case. Therefore we can consider extremal Kähler metrics with singularities as the generalization of constant curvature metrics on Riemann surfaces with conical singularities. In this paper, we study two kinds of singularities: cusp singularities and conical singularities.

Now let Σ be a compact Riemann surface and $\{a_1, a_2, \dots, a_n\}$ be a finite set of Σ . We call a smooth metric g on $\Sigma \setminus \{a_1, a_2, \dots, a_n\}$ an extremal Hermitian metric (v.s.[3]) if g satisfies

$$\Delta_g K + K^2 = C, \tag{1}$$

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where K is the Gauss curvature of g and C is a constant. This condition is equivalent to

$$\frac{\partial K_{,zz}}{\partial \bar{z}} = 0. \quad (2)$$

One can refer to [4] for details. (2) has a special case

$$K_{,zz} = 0. \quad (3)$$

We call a metric an HCMU (the Hessian of the Curvature of the Metric is Umbilical) metric (v.s.[4]) if the metric satisfies (3). Obviously an HCMU metric can be regarded as a direct generalization of an extremal Kähler metric to a punctured Riemann surface. **In the following we always assume that an extremal Hermitian metric or an HCMU metric has finite area and finite Calabi energy**, that is,

$$\int_{\Sigma \setminus \{a_1, a_2, \dots, a_n\}} dg < +\infty, \quad \int_{\Sigma \setminus \{a_1, a_2, \dots, a_n\}} K^2 dg < +\infty. \quad (4)$$

For a general extremal Hermitian metric g on $\Sigma \setminus \{a_1, a_2, \dots, a_n\}$, if g has cusp singularities at a_1, a_2, \dots, a_n , X.X.Chen in [3] proved that g must be an HCMU metric and gave a classification theorem, and if g has conical singularities at a_1, a_2, \dots, a_n and at each singularity the singular angle is less than or equal to $\frac{\pi}{2}$, G.F.Wang and X.H.Zhu in [12] proved that g is also an HCMU metric and gave a classification theorem.

For an HCMU metric g on $\Sigma \setminus \{a_1, a_2, \dots, a_n\}$ which is not a CSC metric, if g has conical singularities at a_1, a_2, \dots, a_n , the first two authors in [7] gave a sufficient and necessary condition for the existence of this kind of metric, that is,

Theorem 1.1 ([7]). *Let M be a compact Riemann surface and p_1, p_2, \dots, p_N be N points on M . Suppose that $\alpha_1, \alpha_2, \dots, \alpha_N$ are N positive real numbers ($\alpha_n \neq 1, n = 1, 2, \dots, N$) and $\alpha_1, \alpha_2, \dots, \alpha_J$ are integers with $\alpha_j \geq 2, j = 1, 2, \dots, J$. Then there exists a normalized HCMU metric g on M such that g has conical singularities at p_n with the angles $2\pi\alpha_n$ ($1 \leq n \leq N$) and p_1, p_2, \dots, p_J are the saddle points of the curvature K , if and only if*

1. $S \triangleq \sum_{j=1}^J \alpha_j + \chi(M) - N \geq 0$,
2. there are S distinct points $\{q_1, q_2, \dots, q_S\} \subset M \setminus \{p_1, p_2, \dots, p_N\}$ such that we can choose L ($0 \leq L \leq N - J$) points in $\{p_{J+1}, p_{J+2}, \dots, p_N\}$ (w.l.o.g. we assume these points are $p_{J+1}, p_{J+2}, \dots, p_{J+L}$), and T ($0 \leq T \leq S$) points in $\{q_1, q_2, \dots, q_S\}$ (w.l.o.g. we assume these points are q_1, q_2, \dots, q_T), to satisfy

$$\text{i) } \alpha_{\max} \triangleq \sum_{l=J+1}^{J+L} \alpha_l + T > \alpha_{\min} \triangleq \sum_{m=J+L+1}^N \alpha_m + S - T > 0,$$

ii) there exists a meromorphic 1-form ω on M satisfying

$$(a) \quad (\omega) = \sum_{j=1}^J (\alpha_j - 1)P_j - \sum_{k=J+1}^N P_k - \sum_{\xi=1}^S Q_\xi,$$

$$(b) \quad \text{Res}_{p_l}(\omega) = \sigma\alpha_l, \quad l = J+1, J+2, \dots, J+L; \quad \text{Res}_{p_m}(\omega) = \sigma\lambda\alpha_m, \quad m = J+L+1, J+L+2, \dots, N; \\ \text{Res}_{q_\mu}(\omega) = \sigma, \quad \mu = 1, 2, \dots, T \quad \text{and} \quad \text{Res}_{q_\nu}(\omega) = \sigma\lambda, \\ \nu = T+1, T+2, \dots, S, \quad \text{where } \lambda = -\frac{\alpha_{\max}}{\alpha_{\min}} \quad \text{and} \quad \sigma = -\frac{(2\lambda+1)^2}{3\lambda(\lambda+1)},$$

(c) $\omega + \bar{\omega}$ is exact on $M \setminus \{p_1, p_2, \dots, p_N, q_1, q_2, \dots, q_S\}$.

In fact from [7] we get any HCMU metric which is not a CSC metric and has conical singularities is determined by the following system:

$$\begin{cases} \frac{dK}{-\frac{1}{3}(K - K_1)(K - K_2)(K + K_1 + K_2)} = \omega + \bar{\omega} \\ g = -\frac{4}{3}(K - K_1)(K - K_2)(K + K_1 + K_2)\omega\bar{\omega} \\ K(p_0) = K_0, \quad K_2 < K_0 < K_1, \quad p_0 \in M \setminus \{\text{zeros and poles of } \omega\}, \end{cases} \quad (5)$$

where $K_1 > 0$ which is the maximum of the Gauss curvature K , $K_1 > K_2 > -\frac{K_1}{2}$ which is the minimum of the Gauss curvature K and ω is a meromorphic 1-form on M with the properties:

1. ω only has simple poles,
2. the residue of ω at each pole is a real number,
3. $\omega + \bar{\omega}$ is exact on $M \setminus \{\text{poles of } \omega\}$.

However (5) does not include the case that an HCMU metric has cusp singularities. Therefore we need to reconsider this case. In this paper we try to give a sufficient and necessary condition for the existence of an HCMU metric which is not a CSC metric and has cusp singularities and conical singularities. That is our main theorem:

Theorem 1.2. *Let Σ be a compact Riemann surface and $p_1, p_2, \dots, p_I, q_1, q_2, \dots, q_J$ be $I + J$ points on Σ ($I > 0, J \geq 0$). Suppose that $\alpha_1, \alpha_2, \dots, \alpha_J$ are J positive real numbers such that $\alpha_j \neq 1$, $j = 1, 2, \dots, J$, and $\alpha_1, \alpha_2, \dots, \alpha_L$ are integers ($L \leq J$). Then there exists an HCMU metric g on $\Sigma \setminus \{p_1, p_2, \dots, p_I, q_1, q_2, \dots, q_J\}$ which is not a CSC metric such that g has cusp singularities at p_i , $i = 1, 2, \dots, I$, has conical singularities at q_j , $j = 1, 2, \dots, J$, with the angle $2\pi\alpha_j$ respectively and q_1, q_2, \dots, q_L are the saddle points of the Gauss curvature K , if and only if*

1. $S := \sum_{l=1}^L \alpha_l - I - J + \chi(\Sigma) \geq 0$,
2. there are S distinct points $\{e_1, e_2, \dots, e_S\} \subset \Sigma \setminus \{p_1, p_2, \dots, p_I, q_1, q_2, \dots, q_J\}$ such that there exists a meromorphic 1-form ω on Σ which satisfies
 - (a) $(\omega) = \sum_{l=1}^L (\alpha_l - 1)Q_l - \sum_{i=1}^I P_i - \sum_{l'=L+1}^J Q_{l'} - \sum_{s=1}^S E_s$;
 - (b) $\text{Res}_{p_i}(\omega) > 0$, $i = 1, 2, \dots, I$,
 $\text{Res}_{q_{l'}}(\omega) = \Lambda\alpha_{l'}$, $l' = L + 1, L + 2, \dots, J$,
 $\text{Res}_{e_s}(\omega) = \Lambda$, $s = 1, 2, \dots, S$,
 where Λ is a negative real number;
 - (c) $\omega + \bar{\omega}$ is exact on $\Sigma \setminus \{p_1, p_2, \dots, p_I, q_1, q_2, \dots, q_J, e_1, e_2, \dots, e_S\}$.

Here we declare that in the following any HCMU metric that we mention is not a CSC metric. In this paper we can get any HCMU metric with cusp singularities and conical

singularities is determined by the following system:

$$\begin{cases} \frac{dK}{-\frac{1}{3}(K-\mu)^2(K+2\mu)} = \omega + \bar{\omega} \\ g = -\frac{4}{3}(K-\mu)^2(K+2\mu)\omega\bar{\omega} \\ K(p_0) = K_0, \mu < K_0 < -2\mu, p_0 \in \Sigma \setminus \{\text{zeros and poles of } \omega\}, \end{cases} \quad (6)$$

where $\mu < 0$ which is the minimum of the Gauss curvature K , -2μ is the maximum of the Gauss curvature K and ω is a meromorphic 1-form on Σ with the same properties as the meromorphic 1-form in (5). From (6) we can obtain that the metric g just has cusp singularities at the poles of ω where the residues of ω are all positive and the Gauss curvature K just has the minimum μ at the cusp singularities of g .

The contents of this paper will be organized as following. In Section 2, we will give definitions of conical singularity and cusp singularity and review some local results about an extremal Hermitian metric around a singularity. Then in Section 3.1, we will prove the necessity of the main theorem. In this section, we first study ∇K of an HCMU metric, then we define the dual 1-form of ∇K as the character 1-form of the metric, and then using the character 1-form we study the properties of g and K at cusp singularities of g , smooth singularities of ∇K and conical singularities of g , finally we give formulas for integrals of K^n over Σ , $n = 0, 1, 2, \dots$. In Section 3.2, we will prove the sufficiency of the main theorem. In this section, first we study the solution of an ODE, then we use the solution to construct an HCMU metric which satisfies the given conditions. In Section 4, we will discuss the existence of the meromorphic 1-form in (5).

2 Definitions of singularities and local behaviors of an extremal Hermitian metric

Definition 2.1 ([11]). *Let X be a Riemann surface, $p \in X$. A conformal metric g on X is said to have a conical singularity at p with the singular angle $2\pi\alpha$ ($\alpha > 0$) if in a neighborhood of p*

$$g = e^{2\varphi}|dz|^2, \quad (7)$$

where z is a local complex coordinate defined in the neighborhood of p with $z(p) = 0$ and

$$\varphi - (\alpha - 1) \ln |z| \quad (8)$$

is continuous at 0.

Remark 2.1. By (7) and (8), in a neighborhood of a conical singularity p , the metric g can also be expressed as

$$g = \frac{h}{|z|^{2-2\alpha}}|dz|^2, \quad (9)$$

where h is a positive continuous function in the neighborhood of p and is smooth in the neighborhood except for the origin. By (8), we have the following limit at p

$$\lim_{z \rightarrow 0} \frac{\varphi + \ln |z|}{\ln |z|} = \alpha. \quad (10)$$

Definition 2.2 ([3]). *Let X be a Riemann surface, $p \in X$. A conformal metric g which has finite area and finite Calabi energy on X is said to have a weak cusp singularity at p if in a neighborhood of p*

$$g = e^{2\varphi}|dz|^2, \quad (11)$$

where z is a local complex coordinate defined in the neighborhood of p with $z(p) = 0$ and

$$\liminf_{r \rightarrow 0} \int_0^{2\pi} r \frac{\partial(\varphi + \ln r)}{\partial r} d\theta = 0 (z = re^{\sqrt{-1}\theta}). \quad (12)$$

Definition 2.3. Let X be a Riemann surface, $p \in X$. A conformal metric g on X is said to have a cusp singularity at p if in a neighborhood of p

$$g = e^{2\varphi} |dz|^2, \quad (13)$$

where z is a local complex coordinate defined in the neighborhood of p with $z(p) = 0$ and

$$\lim_{z \rightarrow 0} \frac{\varphi + \ln |z|}{\ln |z|} = 0. \quad (14)$$

X.X.Chen in [2] proved the following theorem:

Theorem 2.1 ([2]). Let $g = e^{2\varphi} |dz|^2$ be a metric on a punctured disk $D \setminus \{0\}$ with finite area and finite Calabi energy. Define $\phi(r) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(r \cos \theta, r \sin \theta) d\theta$. The following three statements hold true:

- (1) $\lim_{r \rightarrow 0} (\varphi + \ln r) = -\infty$.
- (2) $\lim_{r \rightarrow 0} \phi'(r)r$ exists and is finite.
- (3) There exist a constant $\beta \in (0, 1)$ and two constants C_1 and C_2 such that

$$\frac{1}{\beta} (\phi(r) + \ln r) + C_1 \leq \varphi + \ln r \leq \beta (\phi(r) + \ln r) + C_2.$$

By Theorem 2.1, we have the following corollary:

Corollary 2.1. Let $g = e^{2\varphi} |dz|^2$ be a metric on a punctured disk $D \setminus \{0\}$ with finite area and finite Calabi energy. Then the following two statements are equivalent to each other:

- (1) g has a cusp singularity at 0.
- (2) g has a weak cusp singularity at 0.

Proof. (1) \implies (2): If g has a cusp singularity at 0, that is, $\lim_{r \rightarrow 0} \frac{\varphi + \ln r}{\ln r} = 0$, then

$$\lim_{r \rightarrow 0} \frac{\frac{1}{2\pi} \int_0^{2\pi} (\varphi + \ln r) d\theta}{\ln r} = 0, \text{ i.e. } \lim_{r \rightarrow 0} \frac{\phi(r) + \ln r}{\ln r} = 0.$$

By (1) in Theorem 2.1, $\lim_{r \rightarrow 0} (\phi(r) + \ln r) = -\infty$. By (2) in Theorem 2.1, the limit

$$\lim_{r \rightarrow 0} \frac{(\phi(r) + \ln r)'}{(\ln r)'} = \lim_{r \rightarrow 0} (\phi'(r)r + 1)$$

exists and is finite. Then by L'Hospital's Rule, $\lim_{r \rightarrow 0} (\phi'(r)r + 1) = \lim_{r \rightarrow 0} \frac{\phi(r) + \ln r}{\ln r} = 0$, which implies that 0 is a weak cusp singularity of g .

(2) \implies (1): First by (2) in Theorem 2.1 and L'Hospital's Rule,

$$\lim_{r \rightarrow 0} \frac{\phi(r) + \ln r}{\ln r} = \lim_{r \rightarrow 0} (\phi'(r)r + 1).$$

Since 0 is a weak cusp singularity of g , $\liminf_{r \rightarrow 0} \int_0^{2\pi} r \frac{\partial(\varphi + \ln r)}{\partial r} d\theta = 0$, which means

$$\lim_{r \rightarrow 0} \frac{\phi(r) + \ln r}{\ln r} = \lim_{r \rightarrow 0} (\phi'(r)r + 1) = 0.$$

By (3) in Theorem 2.1, there exist $\beta \in (0, 1)$ and two constants C_1 and C_2 such that

$$\frac{1}{\beta}(\phi(r) + \ln r) + C_1 \leq \varphi + \ln r \leq \beta(\phi(r) + \ln r) + C_2,$$

so

$$\frac{1}{\beta} \frac{\phi(r) + \ln r}{\ln r} + \frac{C_1}{\ln r} \geq \frac{\varphi + \ln r}{\ln r} \geq \beta \frac{\phi(r) + \ln r}{\ln r} + \frac{C_2}{\ln r}.$$

Then $\lim_{r \rightarrow 0} \frac{\varphi + \ln r}{\ln r} = 0$, which shows 0 is a cusp singularity of g . We prove the corollary. \square

Then X.X.Chen in [3] proved the theorem:

Theorem 2.2 ([3]). *Let $g = e^{2\varphi}|dz|^2$ be an extremal Hermitian metric in a punctured disk $D \setminus \{0\}$ with finite area and finite Calabi energy, then*

$$\lim_{z \rightarrow 0} |z|^2 \cdot K \cdot e^{2\varphi} = 0. \quad (15)$$

Further X.X.Chen in [3] proved the following theorem:

Theorem 2.3 ([3]). *Let $g = e^{2\varphi}|dz|^2$ be an extremal Hermitian metric on a punctured disk $D \setminus \{0\}$ with finite area and finite Calabi energy and suppose g has a weak cusp singular point at $z = 0$. Then the following statements hold*

(1) *There exists a constant C_1 such that*

$$|K_{,zz}| \cdot |z| \leq C_1. \quad (16)$$

(2) *There exists a constant C_2 such that*

$$|K_{,z}| \leq C_2 \cdot |z| \cdot e^{2\varphi}. \quad (17)$$

(3) *There exists a negative constant C_3 such that*

$$\lim_{z \rightarrow 0} K = C_3.$$

3 Proof of the main theorem

3.1 Proof of the necessity of the main theorem

Let $\Sigma^* = \Sigma \setminus \{p_1, p_2, \dots, p_I, q_1, q_2, \dots, q_J\}$. Since g is an HCMU metric on Σ^* , then on Σ^*

$$K_{,zz} = 0,$$

which is equivalent to the fact that

$$\nabla K \triangleq \sqrt{-1} K_{,z} \frac{\partial}{\partial z}$$

is a holomorphic vector field on Σ^* . Since an HCMU metric is an extremal Hermitian metric, on Σ^*

$$\Delta_g K + K^2 = C. \quad (18)$$

We will first prove the following proposition:

Proposition 3.1. *There exists a real constant C' such that*

$$-4\sqrt{-1}\nabla K(K) = -\frac{K^3}{3} + CK + C' \quad \text{on } \Sigma^*. \quad (19)$$

Proof. Since g is not a CSC metric, there exists $p \in \Sigma^*$ such that $dK(p) \neq 0$. Let (U, z) be a local complex coordinate chart around p such that U is connected and dK does not vanish on U . Suppose $g = e^{2\varphi}|dz|^2$ on U . Then on U

$$\nabla K = \sqrt{-1}K_{,z} \frac{\partial}{\partial z} = \sqrt{-1}e^{-2\varphi}K_{\bar{z}} \frac{\partial}{\partial z}$$

and

$$4K_{z\bar{z}} = (C - K^2)e^{2\varphi}.$$

Let $F = 4e^{-2\varphi}K_{\bar{z}}$, then F is a holomorphic function on U and does not vanish on U . Therefore

$$4K_{z\bar{z}} = (C - K^2)e^{2\varphi} = (C - K^2)\frac{4K_{\bar{z}}}{F},$$

so

$$K_{z\bar{z}} = \left(\frac{-\frac{K^3}{3} + CK}{F}\right)_{\bar{z}},$$

which means $F_1 \triangleq K_z - \frac{-\frac{K^3}{3} + CK}{F}$ is a holomorphic function on U . Then

$$FK_z = -\frac{K^3}{3} + CK + FF_1.$$

Since $FK_z = 4e^{-2\varphi}K_{\bar{z}}K_z = 4e^{-2\varphi}|K_z|^2$ is a real function, FF_1 which is a holomorphic function on U is a real constant. We denote it by C'_U , so we have

$$-4\sqrt{-1}\nabla K(K) = FK_z = -\frac{K^3}{3} + CK + C'_U \quad \text{on } U.$$

Next let $\mathfrak{S} = \{p \in \Sigma^* | dK(p) = 0\}$, then since g is not a CSC metric \mathfrak{S} is a discrete set of Σ^* . Pick any point $q \in \mathfrak{S}$ and let (V, w) be a local complex coordinate chart around q such that $w(V)$ is a disk and dK does not vanish on $V \setminus \{q\}$. Suppose $g = e^{2\psi}|dw|^2$ on V . Then $G \triangleq 4e^{-2\psi}K_{\bar{w}}$ is a holomorphic function on V and does not vanish on $V \setminus \{q\}$. Similar to above, there is a real constant C'_V such that

$$GK_w = -\frac{K^3}{3} + CK + C'_V \quad \text{on } V \setminus \{q\}. \quad (20)$$

Since both sides of (20) are continuous at q , we have

$$-4\sqrt{-1}\nabla K(K) = -\frac{K^3}{3} + CK + C'_V \quad \text{on } V.$$

Consequently $-4\sqrt{-1}\nabla K(K) + \frac{K^3}{3} - CK$ is a locally constant function on Σ^* . Since Σ^* is connected, $-4\sqrt{-1}\nabla K(K) + \frac{K^3}{3} - CK$ is a global constant on Σ^* . Therefore there is a real constant C' such that

$$-4\sqrt{-1}\nabla K(K) = -\frac{K^3}{3} + CK + C' \quad \text{on } \Sigma^*.$$

We prove the proposition. □

Define $\mathfrak{S} = \{p \in \Sigma^* | dK(p) = 0\}$. Then we have the following proposition:

Proposition 3.2. \mathfrak{S} is finite.

Proof. If \mathfrak{S} is infinite, then \mathfrak{S} has cluster points in Σ since Σ is compact. Suppose $e^* \in \Sigma$ is one of the cluster points of \mathfrak{S} . Obviously $e^* \notin \Sigma^*$, so $e^* = p_i$, some i , $i \in \{1, 2, \dots, I\}$ or $e^* = q_j$, some j , $j \in \{1, 2, \dots, J\}$.

Case 1: $e^* = p_i$, some i , $i \in \{1, 2, \dots, I\}$.

Let (U, z) be a local complex coordinate chart around p_i such that $U \setminus \{p_i\} \subset \Sigma^*$, $z(U)$ is a disk D and $z(p_i) = 0$. Suppose $g = e^{2\varphi}|dz|^2$ on $U \setminus \{p_i\}$. Let $F = 4e^{-2\varphi}K_{\bar{z}}$, then by (2) in Theorem 2.3 F is actually a holomorphic function on D and has a zero at 0. Since p_i is a cluster point of \mathfrak{S} , 0 is a cluster point of the zeros of F . Then $F \equiv 0$ on D , which means $\nabla K \equiv 0$ on Σ^* . It is impossible.

Case 2: $e^* = q_j$, some j , $j \in \{1, 2, \dots, J\}$.

Let (V, w) be a local complex coordinate chart around q_j such that $V \setminus \{q_j\} \subset \Sigma^*$, $w(V)$ is a disk D' and $w(q_j) = 0$. Suppose $g = e^{2\psi}|dw|^2$ on $V \setminus \{q_j\}$. Let $G = 4e^{-2\psi}K_{\bar{w}}$, then G is a holomorphic function on $D' \setminus \{0\}$. Since q_j is a cluster point of \mathfrak{S} , 0 is a cluster point of the zeros of G . Then 0 is a zero of G or an essential singularity of G . If 0 is a zero of G , we can get $G \equiv 0$ on D' . It is impossible. Therefore 0 is an essential singularity of G . By Theorem 2.2

$$\lim_{w \rightarrow 0} |w|^2 \cdot K \cdot e^{2\psi} = 0.$$

On the other hand, since q_j is a conical singularity of g with the angle $2\pi\alpha_j$, by (9) there exists a positive continuous function h on D' such that

$$e^{2\psi} = \frac{h}{|w|^{2-2\alpha_j}} \text{ on } D' \setminus \{0\}.$$

Therefore we have

$$\lim_{w \rightarrow 0} K \cdot |w|^{2\alpha_j} = 0. \quad (21)$$

By Proposition 3.1, (19) holds on $V \setminus \{q_j\}$, that is,

$$\frac{1}{4}|G|^2 \cdot h \cdot |w|^{2\alpha_j-2} = -\frac{K^3}{3} + CK + C' \text{ on } D' \setminus \{0\}.$$

Then by (21), there exists $b \in \mathbb{N}$ such that

$$\lim_{w \rightarrow 0} |G|^2 \cdot |w|^{2b} = 0,$$

which means

$$\lim_{w \rightarrow 0} |G \cdot w^b| = 0.$$

It is a contradiction since 0 is an essential singularity of G .

Consequently we exclude Case 1 and Case 2. That means \mathfrak{S} is finite. We prove the proposition. \square

Now let $\mathfrak{S} = \{e_1, e_2, \dots, e_S\}$ and $\Sigma' = \Sigma^* \setminus \mathfrak{S}$, then ∇K has no zeros on Σ' . We define the dual 1-form of ∇K as the **Character 1-form** of g . Locally let $(\mathfrak{U}, \mathfrak{z})$ be a local complex coordinate chart on Σ' and suppose $g = e^{2u}|d\mathfrak{z}|^2$ on \mathfrak{U} , then

$$\nabla K = \sqrt{-1}e^{-2u}K_{\bar{\mathfrak{z}}}\frac{\partial}{\partial \mathfrak{z}} = \sqrt{-1}\frac{\mathcal{F}}{4}\frac{\partial}{\partial \mathfrak{z}} \text{ on } \mathfrak{U}.$$

Define the **Character 1-form** $\omega = \frac{d\mathfrak{z}}{\mathcal{F}}$ on \mathfrak{U} . Then we have the following proposition:

Proposition 3.3. ω has the following properties:

(1) ω is a meromorphic 1-form on Σ .

(2) On Σ' ,

$$\partial K = \left(-\frac{K^3}{3} + CK + C'\right)\omega \quad (22)$$

or equivalently

$$dK = \left(-\frac{K^3}{3} + CK + C'\right)(\omega + \bar{\omega}). \quad (23)$$

(3) On Σ' ,

$$g = 4\left(-\frac{K^3}{3} + CK + C'\right)\omega\bar{\omega}. \quad (24)$$

Proof. (1): Obviously ω is holomorphic on Σ' . By (2) in Theorem 2.3, each $p_i, i = 1, 2, \dots, I$, is a pole of ω . Since each $e_s, s = 1, 2, \dots, S$, is a zero of ∇K , each $e_s, s = 1, 2, \dots, S$, is a pole of ω . Pick any $q_j, j = 1, 2, \dots, J$, and let (U, z) be a local complex coordinate chart around q_j such that $U \setminus \{q_j\} \subset \Sigma'$, $z(U)$ is a disk D and $z(q_j) = 0$. Suppose $g = e^{2\varphi}|dz|^2$ on $U \setminus \{q_j\}$. Then $F = 4e^{-2\varphi}K_{\bar{z}}$ is a holomorphic function on $D \setminus \{0\}$, so 0 is a removable singularity or a pole or an essential singularity of F . Then we can use the same argument in Case 2 in the proof of Proposition 3.2 to prove 0 is not an essential singularity of F . Hence 0 is a removable singularity or a pole of F , which shows q_j is a regular point or a pole of ω (note $\omega = \frac{dz}{F}$ on $U \setminus \{q_j\}$). Then we finish the proof of (1).

(2), (3): Pick any point $p \in \Sigma'$ and let (V, w) be a local complex coordinate chart around p such that $w(V)$ is a disk D' and $w(p) = 0$. Suppose $g = e^{2\psi}|dw|^2$ on V . Then $G = 4e^{-2\psi}K_{\bar{w}}$ is a nonvanishing holomorphic function on V . Since (19) holds on V , we have

$$GK_w = -\frac{K^3}{3} + CK + C',$$

that is,

$$K_w dw = \left(-\frac{K^3}{3} + CK + C'\right)\frac{dw}{G},$$

which is

$$\partial K|_V = \left(-\frac{K^3}{3} + CK + C'\right)\omega|_V,$$

so we prove (2).

On the other hand, $G = 4e^{-2\psi}K_{\bar{w}}$, that is, $e^{2\psi} = \frac{4K_{\bar{w}}}{G}$, which means

$$g|_V = e^{2\psi}dw d\bar{w} = \frac{dw}{G} 4K_{\bar{w}} d\bar{w} = \omega|_V 4\bar{\omega}|_V = 4\left(-\frac{K^3}{3} + CK + C'\right)\omega|_V \bar{\omega}|_V.$$

Then we prove (3).

□

In the following we will study the roots of $-\frac{K^3}{3} + CK + C' = 0$, the canonical divisor of ω and the residues of ω .

First by (3) in Theorem 2.3, K is continuous at each $p_i, i = 1, 2, \dots, I$, and there are I negative numbers b_1, b_2, \dots, b_I such that $\lim_{p \rightarrow p_i} K(p) = b_i, i = 1, 2, \dots, I$. Then we have the following lemma:

Lemma 3.1. *Each $b_i, i = 1, 2, \dots, I$, is a root of $-\frac{K^3}{3} + CK + C' = 0$.*

Proof. Pick any p_i and let (U, z) be a local complex coordinate chart around p_i such that $U \setminus \{p_i\} \subset \Sigma'$, $z(U)$ is a disk D and $z(p_i) = 0$. Suppose $g = e^{2\varphi}|dz|^2$ on $U \setminus \{p_i\}$. Then $F = 4e^{-2\varphi}K_{\bar{z}}$ is actually a holomorphic function on D and 0 is a zero of F on D . Next on $U \setminus \{p_i\}$ (19) holds, that is,

$$FK_z = \frac{1}{4}|F|^2 e^{2\varphi} = -\frac{K^3}{3} + CK + C'.$$

Since 0 is a zero of F , we assume $F = z\tilde{F}$ on D , where \tilde{F} is a holomorphic function on D . Then on $D \setminus \{0\}$

$$\frac{1}{4}|\tilde{F}|^2 |z|^2 e^{2\varphi} = \frac{1}{4}|\tilde{F}|^2 e^{2(\varphi + \ln r)} = -\frac{K^3}{3} + CK + C', \quad (25)$$

where $r = |z|$. It follows from (1) in Theorem 2.1 that

$$0 = -\frac{b_i^3}{3} + Cb_i + C'.$$

Then we prove the lemma. □

By Lemma 3.1, we get $-\frac{K^3}{3} + CK + C' = 0$ has not a triple root. Otherwise

$$-\frac{K^3}{3} + CK + C' = -\frac{1}{3}(K - a)^3,$$

then $a = 0$, but by Lemma 3.1, each $b_i, i = 1, 2, \dots, I$, which is negative is a root of $-\frac{K^3}{3} + CK + C' = 0$. It is a contradiction. Therefore there are four cases to consider:

$$(C1): -\frac{K^3}{3} + CK + C' = -\frac{1}{3}(K - 2\mu)[(K + \mu)^2 + \mu^*], \text{ where } \mu < 0, \mu^* > 0.$$

$$(C2): -\frac{K^3}{3} + CK + C' = -\frac{1}{3}(K - 2\mu)(K + \mu)^2, \text{ where } \mu < 0.$$

$$(C3): -\frac{K^3}{3} + CK + C' = -\frac{1}{3}(K - \mu)(K - \mu^*)(K + \mu + \mu^*), \text{ where } \mu \neq \mu^*, \mu \neq -(\mu + \mu^*), \mu^* > -(\mu + \mu^*) \text{ and there exists some } b_i \text{ such that } b_i = \mu.$$

$$(C4): -\frac{K^3}{3} + CK + C' = -\frac{1}{3}(K - \mu)^2(K + 2\mu), \text{ where } \mu < 0.$$

Next we will exclude (C1), (C2) and (C3). First we exclude (C1):

Lemma 3.2. *(C1) can not hold.*

Proof. Suppose that (C1) holds. Pick any p_i and let (U, z) be a local complex coordinate chart around p_i such that $U \setminus \{p_i\} \subset \Sigma'$, $z(U)$ is a disk D and $z(p_i) = 0$. Suppose $g = e^{2\varphi}|dz|^2$ on $U \setminus \{p_i\}$. Then $F = 4e^{-2\varphi}K_{\bar{z}}$ is a holomorphic function on D , 0 is a unique zero of F on D and $\omega = \frac{dz}{F}$ on $U \setminus \{p_i\}$. Suppose $\frac{1}{F}$ has the following expression on $D \setminus \{0\}$:

$$\frac{1}{F} = \frac{\lambda_{-k}}{z^k} + \cdots + \frac{\lambda_{-2}}{z^2} + \frac{\lambda_{-1}}{z} + \sum_{m=0}^{\infty} \lambda_m z^m = \frac{\Phi(z)}{z^k}, \quad (26)$$

where $\Phi(z)$ is a holomorphic function on D with $\Phi(0) = \lambda_{-k} \neq 0$. Then

$$\omega = \frac{dz}{F} = \frac{\lambda_{-1}}{z} dz + df_1, \quad (27)$$

where $f_1 = \frac{f_2}{z^{k-1}}$ and f_2 is a holomorphic function on D with $f_2(0) \neq 0$. By Proposition 3.3, on $U \setminus \{p_i\}$ (23) holds. Then we substitute $\omega = \frac{\lambda_{-1}}{z} dz + df_1$ and $-\frac{K^3}{3} + CK + C' = -\frac{1}{3}(K - 2\mu)[(K + \mu)^2 + \mu^*]$ into (23) to get on $U \setminus \{p_i\}$

$$(-3) \frac{dK}{(K - 2\mu)[(K + \mu)^2 + \mu^*]} = \frac{\lambda_{-1} dz}{z} + \frac{\overline{\lambda_{-1}}}{\bar{z}} d\bar{z} + d(2\operatorname{Re}(f_1)). \quad (28)$$

Suppose

$$\frac{1}{(K - 2\mu)[(K + \mu)^2 + \mu^*]} = \frac{\beta_1}{K - 2\mu} + \frac{\beta_2(K + \mu) + \beta_3}{(K + \mu)^2 + \mu^*}.$$

Then

$$\begin{aligned} & \frac{dK}{(K - 2\mu)[(K + \mu)^2 + \mu^*]} = \\ & d\left\{\beta_1 \ln(2\mu - K) + \frac{\beta_2}{2} \ln[(K + \mu)^2 + \mu^*] + \frac{\beta_3}{\sqrt{\mu^*}} \arctan \frac{K + \mu}{\sqrt{\mu^*}}\right\}, \end{aligned} \quad (29)$$

where we use the fact that on $U \setminus \{p_i\}$ (24) holds, so

$$-\frac{K^3}{3} + CK + C' = -\frac{1}{3}(K - 2\mu)[(K + \mu)^2 + \mu^*] > 0 \quad \text{on } U \setminus \{p_i\},$$

which implies $K < 2\mu$ on $U \setminus \{p_i\}$. By (28) and (29), we have $\lambda_{-1} \in \mathbb{R}$. Then we can integrate both sides of (28) to get on $D \setminus \{0\}$

$$\begin{aligned} (-3) \left\{ \beta_1 \ln(2\mu - K) + \frac{\beta_2}{2} \ln[(K + \mu)^2 + \mu^*] + \frac{\beta_3}{\sqrt{\mu^*}} \arctan \frac{K + \mu}{\sqrt{\mu^*}} \right\} = \\ \lambda_{-1} \ln|z|^2 + 2\operatorname{Re}(f_1) + c, \end{aligned} \quad (30)$$

where c is a real constant.

Next since on $U \setminus \{p_i\}$ (24) holds,

$$e^{2\varphi} = -\frac{4}{3}(K - 2\mu)[(K + \mu)^2 + \mu^*] \frac{|\Phi(z)|^2}{|z|^{2k}},$$

that is,

$$\varphi = \frac{1}{2} \left(\ln \frac{2\mu - K}{|z|^{2k}} + \ln \frac{4[(K + \mu)^2 + \mu^*]|\Phi(z)|^2}{3} \right).$$

Then $\lim_{z \rightarrow 0} \frac{\varphi + \ln |z|}{\ln |z|} = 0$ implies

$$\lim_{z \rightarrow 0} \frac{\ln(2\mu - K)}{\ln |z|} = 2(k-1). \quad (31)$$

If $k = 1$, then $\lim_{z \rightarrow 0} \frac{\ln(2\mu - K)}{\ln |z|} = 0$. Divide both sides of (30) by $\ln |z|$, let $z \rightarrow 0$ and take limits. We get

$$\lim_{z \rightarrow 0} \frac{(-\frac{3}{2}\beta_1) \ln(2\mu - K)}{\ln |z|} = \lambda_{-1}, \quad (32)$$

so $\lambda_{-1} = 0$. It is impossible. If $k > 1$, then $\lim_{z \rightarrow 0} \frac{\ln(2\mu - K)}{\ln |z|} = 2(k-1) > 0$. We rewrite (30) to be

$$\begin{aligned} (-3)\{\beta_1 \ln(2\mu - K) + \frac{\beta_2}{2} \ln[(K + \mu)^2 + \mu^*] + \frac{\beta_3}{\sqrt{\mu^*}} \arctan \frac{K + \mu}{\sqrt{\mu^*}}\} = \\ \lambda_{-1} \ln |z|^2 + \frac{f_2}{z^{k-1}} + \frac{\overline{f_2}}{\bar{z}^{k-1}} + c. \end{aligned} \quad (33)$$

Multiply both sides of (33) by z^{k-1} to get

$$\begin{aligned} (-3)\{\beta_1 z^{k-1} \ln(2\mu - K) + \frac{\beta_2}{2} z^{k-1} \ln[(K + \mu)^2 + \mu^*] + \frac{\beta_3}{\sqrt{\mu^*}} z^{k-1} \arctan \frac{K + \mu}{\sqrt{\mu^*}}\} = \\ \lambda_{-1} z^{k-1} \ln |z|^2 + f_2 + \frac{\overline{f_2} z^{k-1}}{\bar{z}^{k-1}} + c z^{k-1}. \end{aligned} \quad (34)$$

Let $z \rightarrow 0$ on both sides of (34) and take limits. Note

$$\lim_{z \rightarrow 0} z^{k-1} \ln(2\mu - K) = \lim_{z \rightarrow 0} \frac{\ln(2\mu - K)}{\ln |z|} z^{k-1} \ln |z| = 0.$$

Therefore we get $\lim_{z \rightarrow 0} \frac{z^{k-1}}{\bar{z}^{k-1}} = A$, $A \neq 0$. It is impossible. Consequently we exclude (C1). \square

Then we exclude (C2):

Lemma 3.3. (C2) can not hold.

Proof. Suppose that (C2) holds. Pick any p_i and let (U, z) be a local complex coordinate chart around p_i such that $U \setminus \{p_i\} \subset \Sigma'$, $z(U)$ is a disk D and $z(p_i) = 0$. Suppose $g = e^{2\varphi} |dz|^2$ on $U \setminus \{p_i\}$. Then $F = 4e^{-2\varphi} K_{\bar{z}}$ is a holomorphic function on D , 0 is a unique zero of F on D and $\omega = \frac{dz}{F}$ on $U \setminus \{p_i\}$. Suppose $\frac{1}{F}$ has the following expression on $D \setminus \{0\}$:

$$\frac{1}{F} = \frac{\lambda_{-k}}{z^k} + \cdots + \frac{\lambda_{-2}}{z^2} + \frac{\lambda_{-1}}{z} + \sum_{m=0}^{\infty} \lambda_m z^m = \frac{\Phi(z)}{z^k}, \quad (35)$$

where $\Phi(z)$ is a holomorphic function on D with $\Phi(0) = \lambda_{-k} \neq 0$. Then

$$\omega = \frac{dz}{F} = \frac{\lambda_{-1}}{z} dz + df_1, \quad (36)$$

where $f_1 = \frac{f_2}{z^{k-1}}$ and f_2 is a holomorphic function on D with $f_2(0) \neq 0$. Similar to the proof of Lemma 3.2, we substitute $\omega = \frac{\lambda_{-1}}{z}dz + df_1$ and $-\frac{K^3}{3} + CK + C' = -\frac{1}{3}(K - 2\mu)(K + \mu)^2$ into (23) to get on $U \setminus \{p_i\}$

$$(-3)\frac{dK}{(K - 2\mu)(K + \mu)^2} = \frac{\lambda_{-1}dz}{z} + \frac{\overline{\lambda_{-1}}}{\bar{z}}d\bar{z} + d(2\operatorname{Re}(f_1)). \quad (37)$$

Suppose

$$\frac{1}{(K - 2\mu)(K + \mu)^2} = \frac{\beta_1}{K - 2\mu} + \frac{\beta_2}{K + \mu} + \frac{\beta_3}{(K + \mu)^2}.$$

Then

$$\begin{aligned} & \frac{dK}{(K - 2\mu)(K + \mu)^2} = \\ & d[\beta_1 \ln(2\mu - K) + \beta_2 \ln(-\mu - K) - \frac{\beta_3}{K + \mu}], \end{aligned} \quad (38)$$

where we use the fact that on $U \setminus \{p_i\}$ (24) holds, so

$$-\frac{K^3}{3} + CK + C' = -\frac{1}{3}(K - 2\mu)(K + \mu)^2 > 0 \quad \text{on } U \setminus \{p_i\},$$

that is, $K < 2\mu < -\mu$. By (37) and (38), we have $\lambda_{-1} \in \mathbb{R}$. Integrate both sides of (37) to get on $D \setminus \{0\}$

$$\begin{aligned} & (-3)[\beta_1 \ln(2\mu - K) + \beta_2 \ln(-\mu - K) - \frac{\beta_3}{K + \mu}] = \\ & \lambda_{-1} \ln |z|^2 + 2\operatorname{Re}(f_1) + c, \end{aligned} \quad (39)$$

where c is a real constant.

On the other hand, since on $U \setminus \{p_i\}$ (24) holds,

$$e^{2\varphi} = -\frac{4}{3}(K - 2\mu)(K + \mu)^2 \frac{|\Phi|^2}{|z|^{2k}},$$

that is,

$$\varphi = \frac{1}{2}(\ln \frac{2\mu - K}{|z|^{2k}} + \ln \frac{4(K + \mu)^2 |\Phi|^2}{3}).$$

Then $\lim_{z \rightarrow 0} \frac{\varphi + \ln |z|}{\ln |z|} = 0$ implies

$$\lim_{z \rightarrow 0} \frac{\ln(2\mu - K)}{\ln |z|} = 2(k - 1). \quad (40)$$

Finally we use the same method in the proof of Lemma 3.2 to get whether $k = 1$ or $k > 1$, there exists a contradiction. Hence we exclude (C2). \square

Finally we exclude (C3):

Lemma 3.4. *(C3) can not hold.*

Proof. Suppose (C3) holds. Fix a point p_i which satisfies $b_i = \mu$. Let (U, z) be a local complex coordinate chart around p_i such that $U \setminus \{p_i\} \subset \Sigma'$, $z(U)$ is a disk D and $z(p_i) = 0$. Suppose $g = e^{2\varphi}|dz|^2$ on $U \setminus \{p_i\}$. Then $F = 4e^{-2\varphi}K_{\bar{z}}$ is a holomorphic function on D , 0 is a unique zero of F on D and $\omega = \frac{dz}{F}$ on $U \setminus \{p_i\}$. Suppose $\frac{1}{F}$ has the following expression on $D \setminus \{0\}$:

$$\frac{1}{F} = \frac{\lambda_{-k}}{z^k} + \cdots + \frac{\lambda_{-2}}{z^2} + \frac{\lambda_{-1}}{z} + \sum_{m=0}^{\infty} \lambda_m z^m = \frac{\Phi(z)}{z^k}, \quad (41)$$

where $\Phi(z)$ is a holomorphic function on D with $\Phi(0) = \lambda_{-k} \neq 0$. Then

$$\omega = \frac{dz}{F} = \frac{\lambda_{-1}}{z} dz + df_1, \quad (42)$$

where $f_1 = \frac{f_2}{z^{k-1}}$ and f_2 is a holomorphic function on D with $f_2(0) \neq 0$. Then we substitute $\omega = \frac{\lambda_{-1}}{z} dz + df_1$ and $-\frac{K^3}{3} + CK + C' = -\frac{1}{3}(K - \mu)(K - \mu^*)(K + \mu + \mu^*)$ into (23) to get on $U \setminus \{p_i\}$

$$(-3) \frac{dK}{(K - \mu)(K - \mu^*)(K + \mu + \mu^*)} = \frac{\lambda_{-1} dz}{z} + \frac{\overline{\lambda_{-1}}}{\bar{z}} d\bar{z} + d(2\operatorname{Re}(f_1)). \quad (43)$$

Suppose

$$\frac{1}{(K - \mu)(K - \mu^*)(K + \mu + \mu^*)} = \frac{\beta_1}{K - \mu} + \frac{\beta_2}{K - \mu^*} + \frac{\beta_3}{K + \mu + \mu^*}.$$

Then

$$\begin{aligned} & \frac{dK}{(K - \mu)(K - \mu^*)(K + \mu + \mu^*)} = \\ & d(\beta_1 \ln |K - \mu| + \beta_2 \ln |K - \mu^*| + \beta_3 \ln |K + \mu + \mu^*|). \end{aligned} \quad (44)$$

Similar to above, $\lambda_{-1} \in \mathbb{R}$. Integrate both sides of (43) to get on $D \setminus \{0\}$

$$\begin{aligned} & (-3)(\beta_1 \ln |K - \mu| + \beta_2 \ln |K - \mu^*| + \beta_3 \ln |K + \mu + \mu^*|) = \\ & \lambda_{-1} \ln |z|^2 + 2\operatorname{Re}(f_1) + c, \end{aligned} \quad (45)$$

where c is a real constant.

On the other hand, since on $U \setminus \{p_i\}$ (24) holds,

$$e^{2\varphi} = \frac{4}{3} |K - \mu| |K - \mu^*| |K + \mu + \mu^*| \frac{|\Phi|^2}{|z|^{2k}},$$

that is,

$$\varphi = \frac{1}{2} \left(\ln \frac{|K - \mu|}{|z|^{2k}} + \ln \frac{4|K - \mu^*| |K + \mu + \mu^*| |\Phi|^2}{3} \right).$$

Then $\lim_{z \rightarrow 0} \frac{\varphi + \ln |z|}{\ln |z|} = 0$ implies

$$\lim_{z \rightarrow 0} \frac{\ln |K - \mu|}{\ln |z|} = 2(k - 1). \quad (46)$$

We can also use the same method in the proof of Lemma 3.2 to exclude (C3). \square

By Lemma 3.1, Lemma 3.2, Lemma 3.3 and Lemma 3.4, we obtain the following theorem:

Theorem 3.1. *There exists a negative number μ such that $b_1 = b_2 = \cdots = b_I = \mu$ and*

$$-\frac{K^3}{3} + CK + C' = -\frac{1}{3}(K - \mu)^2(K + 2\mu).$$

Proof. Since we have excluded cases (C1), (C2) and (C3), (C4) holds, that is, $-\frac{K^3}{3} + CK + C' = -\frac{1}{3}(K - \mu)^2(K + 2\mu)$, where $\mu < 0$. By Lemma 3.1 each b_i , $i = 1, 2, \dots, I$, is a root of $-\frac{K^3}{3} + CK + C' = 0$. Since (C4) holds, $-\frac{K^3}{3} + CK + C' = 0$ has a unique negative root μ . Hence $b_1 = b_2 = \cdots = b_I = \mu$. We prove the theorem. \square

By Theorem 3.1, we get on Σ' $K < -2\mu$. Then there are two possibilities: one is $\forall p \in \Sigma'$, $K(p) < \mu$; the other is $\forall p \in \Sigma'$, $\mu < K(p) < -2\mu$. In the following we will get $\forall p \in \Sigma'$, $\mu < K(p) < -2\mu$. Before the result, we will give the following three lemmas.

Lemma 3.5. *If $\forall p \in \Sigma'$, $K(p) < \mu$, then each p_i , $i = 1, 2, \dots, I$, is a simple pole of ω and the residue of ω at each p_i , $i = 1, 2, \dots, I$, is a negative real number.*

Proof. Pick any p_i and let (U, z) be a local complex coordinate chart around p_i such that $U \setminus \{p_i\} \subset \Sigma'$, $z(U)$ is a disk D and $z(p_i) = 0$. Suppose $g = e^{2\varphi}|dz|^2$ on $U \setminus \{p_i\}$. Then $F = 4e^{-2\varphi}K_{\bar{z}}$ is a holomorphic function on D , 0 is a unique zero of F on D and $\omega = \frac{dz}{F}$ on $U \setminus \{p_i\}$. Suppose $\frac{1}{F}$ has the following expression on $D \setminus \{0\}$:

$$\frac{1}{F} = \frac{\lambda_{-k}}{z^k} + \cdots + \frac{\lambda_{-2}}{z^2} + \frac{\lambda_{-1}}{z} + \sum_{m=0}^{\infty} \lambda_m z^m = \frac{\Phi(z)}{z^k}, \quad (47)$$

where $\Phi(z)$ is a holomorphic function on D with $\Phi(0) = \lambda_{-k} \neq 0$. Then

$$\omega = \frac{dz}{F} = \frac{\lambda_{-1}}{z} dz + df_1, \quad (48)$$

where $f_1 = \frac{f_2}{z^{k-1}}$ and f_2 is a holomorphic function on D with $f_2(0) \neq 0$. Then we substitute $\omega = \frac{\lambda_{-1}}{z} dz + df_1$ and $-\frac{K^3}{3} + CK + C' = -\frac{1}{3}(K - \mu)^2(K + 2\mu)$ into (23) to get on $U \setminus \{p_i\}$

$$(-3) \frac{dK}{(K - \mu)^2(K + 2\mu)} = \frac{\lambda_{-1} dz}{z} + \frac{\overline{\lambda_{-1}}}{\bar{z}} d\bar{z} + d(2\operatorname{Re}(f_1)). \quad (49)$$

Suppose

$$\frac{1}{(K - \mu)^2(K + 2\mu)} = \frac{\beta_1}{K + 2\mu} + \frac{\beta_2}{K - \mu} + \frac{\beta_3}{(K - \mu)^2}.$$

Then $\beta_1 = \frac{1}{9\mu^2}$, $\beta_2 = -\frac{1}{9\mu^2}$, $\beta_3 = \frac{1}{3\mu}$ and on $U \setminus \{p_i\}$

$$\frac{dK}{(K - \mu)^2(K + 2\mu)} = \beta_1 d[\ln(-2\mu - K) - \ln(\mu - K) - \frac{3\mu}{K - \mu}]. \quad (50)$$

We can also get $\lambda_{-1} \in \mathbb{R}$. Integrate both sides of (49) to get on $D \setminus \{0\}$,

$$\begin{aligned} (-3\beta_1)[\ln(-2\mu - K) - \ln(\mu - K) - \frac{3\mu}{K - \mu}] = \\ \lambda_{-1} \ln |z|^2 + 2Re(f_1) + c, \end{aligned} \quad (51)$$

where c is a real constant.

On the other hand,

$$e^{2\varphi} = -\frac{4}{3}(K - \mu)^2(K + 2\mu) \frac{|\Phi|^2}{|z|^{2k}},$$

that is,

$$\varphi = \frac{1}{2} \left(\ln \frac{(K - \mu)^2}{|z|^{2k}} + \ln \frac{4(-2\mu - K)|\Phi|^2}{3} \right).$$

$\lim_{z \rightarrow 0} \frac{\varphi + \ln |z|}{\ln |z|} = 0$ implies

$$\lim_{z \rightarrow 0} \frac{\ln(\mu - K)}{\ln |z|} = k - 1. \quad (52)$$

Suppose that $k > 1$. Multiply both sides of (51) by $\mu - K$ to get

$$\begin{aligned} (\mu - K)(-3\beta_1)[\ln(-2\mu - K) - \ln(\mu - K) + \frac{3\mu}{\mu - K}] = \\ (\mu - K)(\lambda_{-1} \ln |z|^2 + 2Re(f_1) + c). \end{aligned} \quad (53)$$

Then let $z \rightarrow 0$ on both sides of (53) and take limits to get

$$\lim_{z \rightarrow 0} (\mu - K)2Re(f_1) = \lim_{z \rightarrow 0} (\mu - K) \left(\frac{f_2}{z^{k-1}} + \frac{\overline{f_2}}{\overline{z}^{k-1}} \right) = -9\beta_1\mu. \quad (54)$$

Suppose

$$f_2(z) = \nu_0 + zf_3(z),$$

where ν_0 is a nonzero complex number and $f_3(z)$ is a holomorphic function on D . Then

$$\frac{f_2(z)}{z^{k-1}} = \frac{\nu_0}{z^{k-1}} + \frac{f_3(z)}{z^{k-2}} \quad \text{on } D \setminus \{0\}.$$

We claim that

$$\lim_{z \rightarrow 0} \frac{\mu - K}{z^{k-2}} = 0. \quad (55)$$

Since $\lim_{z \rightarrow 0} \frac{\ln(\mu - K)}{\ln |z|} = k - 1$, if z satisfies $0 < |z| < \delta_1$ (δ_1 is small enough), $\frac{\ln(\mu - K)}{\ln |z|} > d > k - 2$. Then $\ln(\mu - K) < d \ln |z|$, that is, $\mu - K < |z|^d$, so

$$0 < \frac{\mu - K}{|z|^{k-2}} < \frac{|z|^d}{|z|^{k-2}}.$$

Hence (55) holds. By (55), we also get

$$\lim_{z \rightarrow 0} \frac{\mu - K}{\overline{z}^{k-2}} = 0. \quad (56)$$

Then by (54), (55) and (56),

$$\lim_{z \rightarrow 0} (\mu - K) \left(\frac{\nu_0}{z^{k-1}} + \frac{\overline{\nu_0}}{\overline{z}^{k-1}} \right) = -9\beta_1\mu. \quad (57)$$

Let $\nu_0 = \nu_1 + \sqrt{-1}\nu_2$ ($\nu_1, \nu_2 \in \mathbb{R}$) and $z = re^{\sqrt{-1}\theta}$ ($0 < r < \delta_2$, $0 \leq \theta < 2\pi$). Then

$$\frac{\nu_0}{z^{k-1}} + \frac{\overline{\nu_0}}{\bar{z}^{k-1}} = \frac{2}{r^{k-1}}[\nu_1 \cos(k-1)\theta + \nu_2 \sin(k-1)\theta]$$

and (57) is

$$\lim_{r \rightarrow 0} (\mu - K) \frac{2}{r^{k-1}} [\nu_1 \cos(k-1)\theta + \nu_2 \sin(k-1)\theta] = -9\beta_1\mu. \quad (58)$$

Now fix a $\theta_0 \in [0, 2\pi)$ such that

$$\begin{cases} \cos(k-1)\theta_0 = \frac{-\nu_2}{\sqrt{\nu_1^2 + \nu_2^2}}, \\ \sin(k-1)\theta_0 = \frac{\nu_1}{\sqrt{\nu_1^2 + \nu_2^2}}. \end{cases}$$

Then

$$\lim_{r \rightarrow 0} [\mu - K(r \cos \theta_0, r \sin \theta_0)] \frac{2}{r^{k-1}} [\nu_1 \cos(k-1)\theta_0 + \nu_2 \sin(k-1)\theta_0] = -9\beta_1\mu \neq 0,$$

however, since $\nu_1 \cos(k-1)\theta_0 + \nu_2 \sin(k-1)\theta_0 = 0$,

$$\lim_{r \rightarrow 0} [\mu - K(r \cos \theta_0, r \sin \theta_0)] \frac{2}{r^{k-1}} [\nu_1 \cos(k-1)\theta_0 + \nu_2 \sin(k-1)\theta_0] = 0.$$

It is a contradiction.

Consequently $k = 1$, that is, p_i is a simple pole of ω . Further divide both sides of (51) by $\ln |z|$, let $z \rightarrow 0$ and take limits to get

$$\lim_{z \rightarrow 0} \frac{\frac{9}{2}\beta_1\mu}{(K - \mu) \ln |z|} = \lambda_{-1}.$$

Therefore $\text{Res}_{p_i}(\omega) = \lambda_{-1} < 0$. We prove the lemma. \square

Lemma 3.6. *If $\forall p \in \Sigma'$, $K(p) < \mu$, then $\mathfrak{S} = \emptyset$.*

Proof. If $\mathfrak{S} \neq \emptyset$, pick a e_s and let (U, z) be a local complex coordinate chart around e_s such that $U \setminus \{e_s\} \subset \Sigma'$, $z(U)$ is a disk D and $z(e_s) = 0$. Suppose $g = e^{2\varphi}|dz|^2$ on U . Then $F = 4e^{-2\varphi}K_{\bar{z}}$ is a holomorphic function on D , 0 is the unique zero of F on D and $\omega = \frac{dz}{F}$ on $U \setminus \{e_s\}$. Suppose $\frac{1}{F}$ has the following expression on $D \setminus \{0\}$:

$$\frac{1}{F} = \frac{\lambda_{-k}}{z^k} + \cdots + \frac{\lambda_{-2}}{z^2} + \frac{\lambda_{-1}}{z} + \sum_{m=0}^{\infty} \lambda_m z^m = \frac{\Phi(z)}{z^k}, \quad (59)$$

where $\Phi(z)$ is a holomorphic function on D with $\Phi(0) = \lambda_{-k} \neq 0$. Then

$$\omega = \frac{\lambda_{-1}}{z} dz + df_1, \quad (60)$$

where $f_1 = \frac{f_2}{z^{k-1}}$ and f_2 is a holomorphic function on D with $f_2(0) \neq 0$.

On one hand, on $D \setminus \{0\}$

$$e^{2\varphi} = -\frac{4}{3}(K - \mu)^2(K + 2\mu)\frac{|\Phi|^2}{|z|^{2k}},$$

then

$$\lim_{z \rightarrow 0} (-2\mu - K)(K - \mu)^2 = 0.$$

Since $\forall p \in \Sigma'$, $K(p) < \mu$, we get

$$\lim_{z \rightarrow 0} K = K(e_s) = \mu.$$

Further we obtain

$$\lim_{z \rightarrow 0} \frac{\mu - K}{|z|^k} = A, \quad A > 0. \quad (61)$$

On the other hand, we substitute $\omega = \frac{\lambda_{-1}}{z}dz + df_1$ and $-\frac{K^3}{3} + CK + C' = -\frac{1}{3}(K - \mu)^2(K + 2\mu)$ into (23) and integrate to get on $D \setminus \{0\}$

$$\begin{aligned} & \left(-\frac{1}{3\mu^2}\right)[\ln(-2\mu - K) - \ln(\mu - K) - \frac{3\mu}{K - \mu}] = \\ & \lambda_{-1} \ln |z|^2 + 2\operatorname{Re}(f_1) + c, \end{aligned} \quad (62)$$

where $\lambda_{-1}, c \in \mathbb{R}$. Then multiply both sides of (62) by $|z|^k$, let $z \rightarrow 0$ and take limits. By (61), the limit of the left side is a nonzero real number. The limit of the right side is 0. It is a contradiction. Therefore we prove the lemma. \square

Lemma 3.7. *If $\forall p \in \Sigma'$, $K(p) < \mu$, then each q_j , $j = 1, 2, \dots, J$, is not a pole of ω .*

Proof. Suppose some q_j is a pole of ω . Let (U, z) be a local complex coordinate chart around q_j such that $U \setminus \{q_j\} \subset \Sigma'$, $z(U)$ is a disk D and $z(q_j) = 0$. Suppose $g = e^{2\varphi}|dz|^2$ on $U \setminus \{q_j\}$. Then $F = 4e^{-2\varphi}K_{\bar{z}}$ is a holomorphic function on D , 0 is the unique zero of F on D and $\omega = \frac{dz}{F}$ on $U \setminus \{q_j\}$. Suppose $\frac{1}{F}$ has the following expression on $D \setminus \{0\}$:

$$\frac{1}{F} = \frac{\lambda_{-k}}{z^k} + \dots + \frac{\lambda_{-2}}{z^2} + \frac{\lambda_{-1}}{z} + \sum_{m=0}^{\infty} \lambda_m z^m = \frac{\Phi(z)}{z^k}, \quad (63)$$

where $\Phi(z)$ is a holomorphic function on D with $\Phi(0) = \lambda_{-k} \neq 0$. Then

$$\omega = \frac{\lambda_{-1}}{z}dz + df_1, \quad (64)$$

where $f_1 = \frac{f_2}{z^{k-1}}$ and f_2 is a holomorphic function on D with $f_2(0) \neq 0$.

Then similar to above, we can get on $D \setminus \{0\}$

$$\begin{aligned} & \left(-\frac{1}{3\mu^2}\right)[\ln(-2\mu - K) - \ln(\mu - K) - \frac{3\mu}{K - \mu}] = \\ & \lambda_{-1} \ln |z|^2 + 2\operatorname{Re}(f_1) + c, \end{aligned} \quad (65)$$

where $\lambda_{-1}, c \in \mathbb{R}$.

On the other hand, since q_j is a conical singularity of g with the singular angle $2\pi\alpha_j$, we suppose

$$g = e^{2\varphi}|dz|^2 = \frac{h}{|z|^{2-2\alpha_j}}|dz|^2 \quad \text{on } D \setminus \{0\},$$

where h is a positive continuous function on D . By (24),

$$\frac{h}{|z|^{2-2\alpha_j}} = -\frac{4}{3}(K - \mu)^2(K + 2\mu)\frac{|\Phi|^2}{|z|^{2k}}.$$

Then

$$\lim_{z \rightarrow 0} (-2\mu - K)(K - \mu)^2 = 0.$$

Since $\forall p \in \Sigma', K(p) < \mu$,

$$\lim_{z \rightarrow 0} K = \mu.$$

Further we get

$$\lim_{z \rightarrow 0} \frac{\mu - K}{|z|^{k-1+\alpha_j}} = A, \quad A > 0.$$

Then multiply both sides of (65) by $|z|^{k-1+\alpha_j}$, let $z \rightarrow 0$ and take limits. The limit of the left side is a nonzero real number. The limit of the right side is 0. It is a contradiction. Therefore we prove the lemma. \square

Now we can get the following theorem:

Theorem 3.2. $\forall p \in \Sigma', \mu < K(p) < -2\mu$.

Proof. Otherwise $\forall p \in \Sigma', K(p) < \mu$. Then by Lemma 3.5, each p_i is a simple pole of ω and the residue of ω at each p_i is a negative real number. Since ω is a meromorphic 1-form on Σ , the sum of the residues of ω is zero. That means ω must have other poles besides p_1, p_2, \dots, p_I . Obviously the set of these poles of ω besides p_1, p_2, \dots, p_I is a subset of $\{e_1, e_2, \dots, e_S, q_1, q_2, \dots, q_J\}$. By Lemma 3.6, $\mathfrak{S} = \emptyset$ and by Lemma 3.7, each q_j is not a pole of ω . It is a contradiction. Hence $\forall p \in \Sigma', \mu < K(p) < -2\mu$. We prove the theorem. \square

Next we will get a theorem about the cusp singularities:

Theorem 3.3. *Each $p_i, i = 1, 2, \dots, I$, is a simple pole of ω and the residue of ω at each p_i is a positive real number.*

Proof. Using the similar argument in the proof of Lemma 3.5 (the only difference is in the proof of Lemma 3.5 $K < \mu$, but here $K > \mu$), we can prove this theorem. \square

Then a theorem about \mathfrak{S} will be obtained.

Theorem 3.4. *If $\mathfrak{S} \neq \emptyset$, then each $e_s, s = 1, 2, \dots, S$, is a simple pole of ω , the residue of ω at each $e_s, s = 1, 2, \dots, S$, is $-\frac{1}{3\mu^2}$ and $K(e_s) = -2\mu, s = 1, 2, \dots, S$.*

Proof. Suppose that $\mathfrak{S} \neq \emptyset$. Pick any e_s and let (U, z) be a local complex coordinate chart around e_s such that $U \setminus \{e_s\} \subset \Sigma', z(U)$ is a disk D and $z(e_s) = 0$. Suppose $g = e^{2\varphi}|dz|^2$ on U .

Then $F = 4e^{-2\varphi}K_{\bar{z}}$ is a holomorphic function on D , 0 is the unique zero of F on D and $\omega = \frac{dz}{F}$ on $U \setminus \{e_s\}$. Suppose $\frac{1}{F}$ has the following expression on $D \setminus \{0\}$:

$$\frac{1}{F} = \frac{\lambda_{-k}}{z^k} + \cdots + \frac{\lambda_{-2}}{z^2} + \frac{\lambda_{-1}}{z} + \sum_{m=0}^{\infty} \lambda_m z^m = \frac{\Phi(z)}{z^k}, \quad (66)$$

where $\Phi(z)$ is a holomorphic function on D with $\Phi(0) = \lambda_{-k} \neq 0$. Then

$$\omega = \frac{\lambda_{-1}}{z} dz + df_1, \quad (67)$$

where $f_1 = \frac{f_2}{z^{k-1}}$ and f_2 is a holomorphic function on D with $f_2(0) \neq 0$.

Then first on $D \setminus \{0\}$

$$\begin{aligned} (-\frac{1}{3\mu^2})[\ln(-2\mu - K) - \ln(K - \mu) - \frac{3\mu}{K - \mu}] = \\ \lambda_{-1} \ln |z|^2 + 2\operatorname{Re}(f_1) + c, \end{aligned} \quad (68)$$

where $\lambda_{-1}, c \in \mathbb{R}$.

On the other hand, on $D \setminus \{0\}$

$$e^{2\varphi} = -\frac{4}{3}(K - \mu)^2(K + 2\mu) \frac{|\Phi|^2}{|z|^{2k}}.$$

Then

$$\lim_{z \rightarrow 0} (K - \mu)^2(-2\mu - K) = 0.$$

Since K is continuous at e_s , $K(e_s) = \mu$ or $K(e_s) = -2\mu$. If $K(e_s) = \mu$, then

$$\lim_{z \rightarrow 0} \frac{K - \mu}{|z|^k} = A_1, \quad A_1 > 0.$$

Multiply both sides of (68) by $|z|^k$, let $z \rightarrow 0$ and take limits. The limit of the left side is a nonzero real number. The limit of the right side is 0. It is a contradiction. Therefore $K(e_s) = -2\mu$ and

$$\lim_{z \rightarrow 0} \frac{-2\mu - K}{|z|^{2k}} = A_2, \quad A_2 > 0.$$

Then

$$\lim_{z \rightarrow 0} \frac{\ln(-2\mu - K)}{\ln |z|} = 2k. \quad (69)$$

If $k > 1$, multiply both sides of (68) by z^{k-1} to get

$$\begin{aligned} z^{k-1}(-\frac{1}{3\mu^2})[\ln(-2\mu - K) - \ln(K - \mu) - \frac{3\mu}{K - \mu}] = \\ z^{k-1}\lambda_{-1} \ln |z|^2 + f_2 + \frac{z^{k-1}}{f_2 \bar{z}^{k-1}} + cz^{k-1}. \end{aligned} \quad (70)$$

Then let $z \rightarrow 0$ on both sides of (70) and take limits. By (69), we get

$$\lim_{z \rightarrow 0} \frac{z^{k-1}}{\bar{z}^{k-1}} = A_3, \quad A_3 \neq 0.$$

It is impossible. Hence $k = 1$, that is, e_s is a simple pole of ω . Then divide both sides of (68) by $\ln |z|$, let $z \rightarrow 0$ and take limits to get

$$-\frac{1}{3\mu^2} \lim_{z \rightarrow 0} \frac{\ln(-2\mu - K)}{\ln |z|} = 2\lambda_{-1}.$$

By (69), $\lambda_{-1} = \text{Res}_{e_s}(\omega) = -\frac{1}{3\mu^2}$. Therefore we prove the theorem. \square

Next we will consider the conical singularities of g, q_1, q_2, \dots, q_J . First we get the following result: if the singular angle of g at q_j is 2π , then q_j is a regular point of g . Before the result, we will give a lemma:

Lemma 3.8. *Suppose that Ω is a domain in \mathbb{R}^N and $0 \in \Omega$. Let f be a continuous function on Ω and $f|_{\Omega \setminus \{0\}} \in C^1(\Omega \setminus \{0\})$. If there exist $g_1, g_2, \dots, g_N \in C^0(\Omega)$ such that $\frac{\partial f}{\partial x_\nu} = g_\nu$ holds on $\Omega \setminus \{0\}$, $\forall \nu, \nu = 1, 2, \dots, N$, then $f \in C^1(\Omega)$.*

Proof. In fact we only need to prove that f has partial derivatives at 0 and $\frac{\partial f}{\partial x_\nu}(0) = g_\nu(0), \forall \nu, \nu = 1, 2, \dots, N$.

Pick any ν . Suppose that $(\underbrace{0, \dots, t, \dots, 0}_\nu) \in \Omega$ as $t \in [-\Delta, \Delta] (\Delta > 0)$. Define $\Gamma(t) = f(\underbrace{0, \dots, t, \dots, 0}_\nu), t \in [0, \Delta]$. Then $\Gamma(t)$ is continuous on $[0, \Delta]$ and $\Gamma'(t) = g_\nu(\underbrace{0, \dots, t, \dots, 0}_\nu), \forall t \in (0, \Delta)$. Then by Newton-Leibnitz formula

$$\int_0^t g_\nu(\underbrace{0, \dots, \tau, \dots, 0}_\nu) d\tau = \Gamma(t) - \Gamma(0), \quad 0 < t < \Delta.$$

Hence

$$\lim_{t \rightarrow 0^+} \frac{\Gamma(t) - \Gamma(0)}{t} = \lim_{t \rightarrow 0^+} \frac{f(\underbrace{0, \dots, t, \dots, 0}_\nu) - f(0)}{t} = g_\nu(0).$$

Similarly,

$$\lim_{t \rightarrow 0^-} \frac{f(\underbrace{0, \dots, t, \dots, 0}_\nu) - f(0)}{t} = g_\nu(0).$$

Therefore

$$\frac{\partial f}{\partial x_\nu}(0) = g_\nu(0).$$

Then we prove the lemma. \square

Proposition 3.4. *If the singular angle of g at q_j is 2π , then q_j is a regular point of g .*

Proof. Let (U, z) be a local complex coordinate chart around q_j such that $U \setminus \{q_j\} \subset \Sigma'$, $z(U)$ is a disk D and $z(q_j) = 0$. Since q_j is a conical singularity of g with the angle 2π , we suppose $g = h|dz|^2$ on $U \setminus \{q_j\}$, where h is a positive continuous function on U and is smooth on $U \setminus \{q_j\}$.

Then $F = \frac{4K_{\bar{z}}}{h}$ is a nonvanishing holomorphic function on $U \setminus \{q_j\}$ and $\omega = \frac{dz}{F}$ on $U \setminus \{q_j\}$. By (24), on $D \setminus \{0\}$

$$h = -\frac{4}{3}(K - \mu)^2(K + 2\mu)\frac{1}{|F|^2}.$$

By Theorem 3.2, $-(K - \mu)^2(K + 2\mu)$ is bounded on $D \setminus \{0\}$, so q_j is not a zero of ω .

If q_j is a pole of ω , suppose $\frac{1}{F}$ has the following expression on $D \setminus \{0\}$,

$$\frac{1}{F} = \frac{\lambda_{-k}}{z^k} + \cdots + \frac{\lambda_{-2}}{z^2} + \frac{\lambda_{-1}}{z} + \sum_{m=0}^{\infty} \lambda_m z^m = \frac{\Phi(z)}{z^k},$$

where $\Phi(z)$ is a holomorphic function on D with $\Phi(0) = \lambda_{-k} \neq 0$. Therefore

$$\omega = \frac{\lambda_{-1}}{z} dz + df_1,$$

where $f_1 = \frac{f_2}{z^{k-1}}$ and f_2 is a holomorphic function on D with $f_2(0) \neq 0$. Then we get on $D \setminus \{0\}$

$$\begin{aligned} & \left(-\frac{1}{3\mu^2}\right)[\ln(-2\mu - K) - \ln(K - \mu) - \frac{3\mu}{K - \mu}] = \\ & \lambda_{-1} \ln|z|^2 + 2\operatorname{Re}(f_1) + c_1, \end{aligned} \quad (71)$$

where $\lambda_{-1}, c_1 \in \mathbb{R}$. On the other hand, on $D \setminus \{0\}$

$$h = -\frac{4}{3}(K - \mu)^2(K + 2\mu) \frac{|\Phi|^2}{|z|^{2k}},$$

so

$$\lim_{z \rightarrow 0} (-2\mu - K)(K - \mu)^2 = 0.$$

If $\limsup_{z \rightarrow 0} K = -2\mu$ and $\liminf_{z \rightarrow 0} K = \mu$, then there exist two sequences $\{x_\ell\}, \{x'_\ell\} \subset D \setminus \{0\}$ such that $x_\ell \rightarrow 0, x'_\ell \rightarrow 0$ as $\ell \rightarrow \infty$ and $\lim_{\ell \rightarrow \infty} K(x_\ell) = -2\mu, \lim_{\ell \rightarrow \infty} K(x'_\ell) = \mu$. Pick any $\hat{\mu}$ such that $\mu < \hat{\mu} < -2\mu$. As ℓ is big enough, $K(x_\ell) > \hat{\mu}$ and $K(x'_\ell) < \hat{\mu}$. Then there exists y_ℓ which satisfies $\min\{|x_\ell|, |x'_\ell|\} \leq |y_\ell| \leq \max\{|x_\ell|, |x'_\ell|\}$ such that $K(y_\ell) = \hat{\mu}$. Obviously $y_\ell \rightarrow 0$ as $\ell \rightarrow \infty$, so

$$\lim_{\ell \rightarrow \infty} (-2\mu - K(y_\ell))(K(y_\ell) - \mu)^2 = 0,$$

which means $(-2\mu - \hat{\mu})(\hat{\mu} - \mu)^2 = 0$. It is impossible. Therefore $\lim_{z \rightarrow 0} K = -2\mu$ or $\lim_{z \rightarrow 0} K = \mu$. If $\lim_{z \rightarrow 0} K = \mu$, we can use the same method in the proof of Theorem 3.4 to get a contradiction. Then $\lim_{z \rightarrow 0} K = -2\mu$, which means K is continuous on D . Further

$$\lim_{z \rightarrow 0} \frac{-2\mu - K}{|z|^{2k}} = A, \quad A > 0,$$

which implies

$$\lim_{z \rightarrow 0} \frac{\ln(-2\mu - K)}{\ln|z|} = 2k.$$

If $k > 1$, we can also use the same method in the proof of Theorem 3.4 to get a contradiction.

Hence $k = 1$. Then divide both sides of (71) by $\ln|z|$, let $z \rightarrow 0$ and take limits to get $\lambda_{-1} = \frac{-1}{3\mu^2}$.

Next by (71), on $D \setminus \{0\}$

$$-2\mu - K = |z|^2(K - \mu)e^{\frac{3\mu}{K - \mu}} f_3,$$

where f_3 is a positive smooth function on D . Therefore on $D \setminus \{0\}$

$$\begin{aligned} h &= \frac{4}{3}(K - \mu)^2 |z|^2 (K - \mu) e^{\frac{3\mu}{K-\mu}} f_3 \frac{|\Phi|^2}{|z|^2} \\ &= (K - \mu)^3 e^{\frac{3\mu}{K-\mu}} f_4, \end{aligned}$$

where f_4 is a positive smooth function on D . Then consider the following system of equations on $D \setminus \{0\}$:

$$\begin{cases} K_{\bar{z}} = \frac{1}{4} F h, \\ h = (K - \mu)^3 e^{\frac{3\mu}{K-\mu}} f_4. \end{cases} \quad (72)$$

By Lemma 3.8 and the first equation of (72), $K \in C^1(D)$. Then using the bootstrap technique to (72), we get $K \in C^\infty(D)$. Finally by the second equation of (72), $h \in C^\infty(D)$, which means q_j is a regular point of g .

If q_j is a regular point of ω (neither a pole nor a zero of ω), then $\omega = f'_5(z)dz = df_5$, where f_5 is a holomorphic function on D . Therefore on $D \setminus \{0\}$

$$-\frac{1}{3\mu^2} [\ln(-2\mu - K) - \ln(K - \mu) - \frac{3\mu}{K - \mu}] = f_5 + \overline{f_5} + c_2, \quad (73)$$

where $c_2 \in \mathbb{R}$. Let

$$\sigma(t) = \ln(-2\mu - t) - \ln(t - \mu) - \frac{3\mu}{t - \mu}, \quad t \in (\mu, -2\mu).$$

Then $\sigma(t)$ has the following property: $\forall x \in \mathbb{R}, \exists! t \in (\mu, -2\mu)$, s.t. $\sigma(t) = x$. The reasons for the existence are the continuity of $\sigma(t)$ and

$$\lim_{t \rightarrow (-2\mu)^-} \sigma(t) = -\infty, \quad \lim_{t \rightarrow \mu^+} \sigma(t) = +\infty.$$

The reason for the uniqueness is since

$$\sigma'(t) = \frac{1}{t + 2\mu} - \frac{1}{t - \mu} + \frac{3\mu}{(t - \mu)^2} = \frac{9\mu^2}{(t - \mu)^2(t + 2\mu)},$$

$\sigma'(t) \neq 0, \forall t \in (\mu, -2\mu)$. By the property of $\sigma(t)$, we can define a function T on D such that $\mu < T < -2\mu$ and $\sigma(T) = (-3\mu^2)(f_5 + \overline{f_5} + c_2)$. Then by the implicit theorem, $T \in C^\infty(D)$. By (73), $K = T$ on $D \setminus \{0\}$, so K can be smoothly extended to q_j . By (24), on $D \setminus \{0\}$

$$h = -\frac{4}{3}(K - \mu)^2(K + 2\mu)|f'_5(z)|^2.$$

Since K is smooth on D and q_j is a regular point of ω , h is smooth on D , which also means q_j is a regular point of g . Then we prove the proposition. \square

Remark 3.1. By Proposition 3.4, we suppose that the singular angle of g at each q_j , $j = 1, 2, \dots, J$, is not 2π . That is why we suppose $\alpha_j \neq 1$, $j = 1, 2, \dots, J$, in Theorem 1.2.

Next we will get the following theorem for the conical singularities q_1, q_2, \dots, q_J :

Theorem 3.5. Each q_j , $j = 1, 2, \dots, J$, is a pole or a zero of ω . If q_j is a zero of ω , then α_j must be an integer, the order of ω at q_j is $\alpha_j - 1$, K can be smoothly extended to q_j with $\mu < K(q_j) < -2\mu$ and $dK(q_j) = 0$. If q_j is a pole of ω , then q_j is a simple pole of ω , the residue of ω at q_j is $-\frac{\alpha_j}{3\mu^2}$ and K can be continuously extended to q_j with $K(q_j) = -2\mu$.

Proof. Suppose that q_j is a regular point of ω . Let (W, ξ) be a local complex coordinate chart around q_j such that $W \setminus \{q_j\} \subset \Sigma'$, $\xi(W)$ is a disk \tilde{D} and $\xi(q_j) = 0$. Assume $\omega = \rho(\xi)d\xi$ on W , where $\rho(\xi)$ is a holomorphic function on \tilde{D} with $\rho(0) \neq 0$. Then there exists a positive continuous function \tilde{h} on \tilde{D} such that on $\tilde{D} \setminus \{0\}$

$$\frac{\tilde{h}}{|\xi|^{2-2\alpha_j}} = -\frac{4}{3}(K - \mu)^2(K + 2\mu)|\rho(\xi)|^2. \quad (74)$$

Then we can also use the argument in the proof of Proposition 3.4 to get K can be smoothly extended to q_j with $\mu < K(q_j) < -2\mu$. If $\alpha_j < 1$, let $\xi \rightarrow 0$ and take limits on both sides of (74). The limit of the left side is $+\infty$ and the limit of the right side is a nonzero real number. It is a contradiction. If $\alpha_j > 1$, let $\xi \rightarrow 0$ and take limits on both sides of (74). The limit of the left side is 0 and the limit of the right side is a nonzero real number. It is also a contradiction. Hence q_j is not a regular point of ω .

Suppose q_j is a zero of ω . Let (Y, ζ) be a local complex coordinate chart around q_j such that $Y \setminus \{q_j\} \subset \Sigma'$, $\zeta(Y)$ is a disk \hat{D} and $\zeta(q_j) = 0$. Assume on Y

$$\omega = \zeta^\gamma H_1 d\zeta = dH_2,$$

where γ is the order of ω at q_j , H_1 is a holomorphic function on \hat{D} with $H_1(0) \neq 0$ and H_2 is a holomorphic function on \hat{D} . Then also by the argument in the proof of Proposition 3.4, we have K can be smoothly extended to q_j with $\mu < K(q_j) < -2\mu$. By (23), $dK(q_j) = 0$. By (24), there exists a positive continuous function \hat{h} on \hat{D} such that on $\hat{D} \setminus \{0\}$

$$\frac{\hat{h}}{|\zeta|^{2-2\alpha_j}} = -\frac{4}{3}(K - \mu)^2(K + 2\mu)|\zeta|^{2\gamma}|H_1|^2.$$

Therefore $\gamma = \alpha_j - 1$.

Suppose q_j is a pole of ω . Let (U, z) be a local complex coordinate chart around q_j such that $U \setminus \{q_j\} \subset \Sigma'$, $z(U)$ is a disk D and $z(q_j) = 0$. Suppose $g = e^{2\varphi}|dz|^2$ on $U \setminus \{q_j\}$. Then $F = 4e^{-2\varphi}K_{\bar{z}}$ is a holomorphic function on D , 0 is the unique zero of F on D and $\omega = \frac{dz}{F}$ on $U \setminus \{q_j\}$. Suppose $\frac{1}{F}$ has the following expression on $D \setminus \{0\}$:

$$\frac{1}{F} = \frac{\lambda_{-k}}{z^k} + \cdots + \frac{\lambda_{-2}}{z^2} + \frac{\lambda_{-1}}{z} + \sum_{m=0}^{\infty} \lambda_m z^m = \frac{\Phi(z)}{z^k}, \quad (75)$$

where $\Phi(z)$ is a holomorphic function on D with $\Phi(0) = \lambda_{-k} \neq 0$. Then

$$\omega = \frac{\lambda_{-1}}{z} dz + df_1, \quad (76)$$

where $f_1 = \frac{f_2}{z^{k-1}}$ and f_2 is a holomorphic function on D with $f_2(0) \neq 0$. Then on $D \setminus \{0\}$

$$\begin{aligned} \left(-\frac{1}{3\mu^2}\right)[\ln(-2\mu - K) - \ln(K - \mu) - \frac{3\mu}{K - \mu}] = \\ \lambda_{-1} \ln |z|^2 + 2\operatorname{Re}(f_1) + c. \end{aligned} \quad (77)$$

where $\lambda_{-1}, c \in \mathbb{R}$.

On the other hand, there exists a positive continuous function h on D such that on $D \setminus \{0\}$

$$\frac{h}{|z|^{2-2\alpha_j}} = -\frac{4}{3}(K - \mu)^2(K + 2\mu)\frac{|\Phi|^2}{|z|^{2k}}.$$

Then

$$\lim_{z \rightarrow 0} (K - \mu)^2(K + 2\mu) = 0.$$

By the same argument in the proof of Proposition 3.4,

$$\lim_{z \rightarrow 0} K = \mu \quad \text{or} \quad \lim_{z \rightarrow 0} K = -2\mu.$$

If $\lim_{z \rightarrow 0} K = \mu$, then

$$\lim_{z \rightarrow 0} \frac{K - \mu}{|z|^{k+\alpha_j-1}} = A_1, \quad A_1 > 0.$$

Multiply both sides of (77) by $|z|^{k+\alpha_j-1}$, let $z \rightarrow 0$ and take limits. The limit of the left side is a nonzero real number and the limit of the right side is 0. It is a contradiction. Hence we have $\lim_{z \rightarrow 0} K = -2\mu$. Then

$$\lim_{z \rightarrow 0} \frac{-2\mu - K}{|z|^{2k+2\alpha_j-2}} = A_2, \quad A_2 > 0,$$

which implies

$$\lim_{z \rightarrow 0} \frac{\ln(-2\mu - K)}{\ln |z|} = 2k + 2\alpha_j - 2.$$

If $k > 1$, we can also use the argument in the proof of Theorem 3.4 to get a contradiction. Therefore $k = 1$, that is, q_j is a simple pole of ω . Then divide both sides of (77) by $\ln |z|$, let $z \rightarrow 0$ and take limits. The limit of the left side is $-\frac{2\alpha_j}{3\mu^2}$ and the limit of the right side is $2\lambda_{-1}$, so

$$Res_{q_j}(\omega) = \lambda_{-1} = -\frac{\alpha_j}{3\mu^2}.$$

Then we prove the theorem. □

Therefore we finish the proof of the necessity of Theorem 1.2.

Remark 3.2. By (3) in Theorem 2.3 and Theorem 3.5, K is a continuous function on Σ . By the assumption in Theorem 1.2 and Theorem 3.5, q_1, q_2, \dots, q_L which are the saddle points of K are the zeros of ω and $q_{L+1}, q_{L+2}, \dots, q_J$ are the poles of ω .

In the following, we will give formulas for

$$\mathcal{C}_n = \int_{\Sigma'} K^n dg, \quad n = 0, 1, 2, \dots$$

Obviously, \mathcal{C}_0 is the area of g , \mathcal{C}_1 is related to the generalized Gauss- Bonnet formula and \mathcal{C}_2 is

the Calabi energy of g . First

$$\begin{aligned}
\mathcal{C}_n &= \int_{\Sigma'} K^n dg \\
&= \frac{\sqrt{-1}}{2} \int_{\Sigma'} K^n \frac{-4}{3} (K - \mu)^2 (K + 2\mu) \omega \wedge \bar{\omega} \\
&= 2\sqrt{-1} \int_{\Sigma'} K^n \partial K \wedge \bar{\omega} \\
&= \frac{2\sqrt{-1}}{n+1} \int_{\Sigma'} d(K^{n+1} \bar{\omega}) \\
&= \lim_{\varepsilon \rightarrow 0} \frac{2\sqrt{-1}}{n+1} \int_{\Sigma \setminus (\cup_{i=1}^I D_\varepsilon(p_i) \cup \cup_{s=1}^S D_\varepsilon(e_s) \cup \cup_{j=1}^J D_\varepsilon(q_j))} d(K^{n+1} \bar{\omega}),
\end{aligned}$$

where $D_\varepsilon(p_i)(D_\varepsilon(e_s), D_\varepsilon(q_j))$ is a coordinate disk around $p_i(e_s, q_j)$ with the center $p_i(e_s, q_j)$ and the radius ε . By the Stokes formula,

$$\begin{aligned}
&\int_{\Sigma \setminus (\cup_{i=1}^I D_\varepsilon(p_i) \cup \cup_{s=1}^S D_\varepsilon(e_s) \cup \cup_{j=1}^J D_\varepsilon(q_j))} d(K^{n+1} \bar{\omega}) = \\
&-\sum_{i=1}^I \oint_{\partial D_\varepsilon(p_i)} K^{n+1} \bar{\omega} - \sum_{s=1}^S \oint_{\partial D_\varepsilon(e_s)} K^{n+1} \bar{\omega} - \sum_{j=1}^J \oint_{\partial D_\varepsilon(q_j)} K^{n+1} \bar{\omega},
\end{aligned}$$

where the directions of the integrations are anticlockwise. Consider

$$\lim_{\varepsilon \rightarrow 0} \oint_{\partial D_\varepsilon(q_j)} K^{n+1} \bar{\omega}.$$

If q_j is a zero of ω , suppose on $D_{\varepsilon_0}(q_j)$

$$\omega = \rho_1(z) dz,$$

where $\rho_1(z)$ is a holomorphic function on the coordinate disk $D_{\varepsilon_0}(q_j)$. Then $\forall \varepsilon, 0 < \varepsilon < \varepsilon_0$,

$$\oint_{\partial D_\varepsilon(q_j)} K^{n+1} \bar{\omega} = (-\sqrt{-1})\varepsilon \int_0^{2\pi} K^{n+1} \bar{\rho}_1 e^{-\sqrt{-1}\theta} d\theta,$$

where $z = re^{\sqrt{-1}\theta}$ on $D_{\varepsilon_0}(q_j)$. Since K and $\bar{\rho}_1$ are bounded around q_j ,

$$\lim_{\varepsilon \rightarrow 0} \oint_{\partial D_\varepsilon(q_j)} K^{n+1} \bar{\omega} = 0.$$

If q_j is a pole of ω , suppose on $D_{\varepsilon_1}(q_j) \setminus \{0\}$

$$\omega = \frac{\lambda_{-1}}{z} dz + \rho_2(z) dz,$$

where $\lambda_{-1} = \text{Res}_{q_j}(\omega)$ and $\rho_2(z)$ is a holomorphic function on the coordinate disk $D_{\varepsilon_1}(q_j)$. Then $\forall \varepsilon, 0 < \varepsilon < \varepsilon_1$,

$$\begin{aligned}
\oint_{\partial D_\varepsilon(q_j)} K^{n+1} \bar{\omega} &= \oint_{\partial D_\varepsilon(q_j)} K^{n+1} \left(\frac{\lambda_{-1}}{\bar{z}} d\bar{z} + \overline{\rho_2(z)} d\bar{z} \right) \\
&= (-\sqrt{-1}\lambda_{-1}) \int_0^{2\pi} K^{n+1} d\theta + \oint_{\partial D_\varepsilon(q_j)} K^{n+1} \overline{\rho_2} d\bar{z}.
\end{aligned}$$

Since

$$\lim_{\varepsilon \rightarrow 0} \int_0^{2\pi} K^{n+1} d\theta = 2\pi K(q_j)^{n+1} \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \oint_{\partial D_\varepsilon(q_j)} K^{n+1} \overline{\rho_2} d\bar{z} = 0,$$

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \oint_{\partial D_\varepsilon(q_j)} K^{n+1} \bar{\omega} &= (-\sqrt{-1} \lambda_{-1}) 2\pi K(q_j)^{n+1} \\ &= (-2\pi \sqrt{-1}) \text{Res}_{q_j}(\omega) (-2\mu)^{n+1}. \end{aligned}$$

Similarly,

$$\lim_{\varepsilon \rightarrow 0} \oint_{\partial D_\varepsilon(e_s)} K^{n+1} \bar{\omega} = (-2\pi \sqrt{-1}) \text{Res}_{e_s}(\omega) (-2\mu)^{n+1},$$

$$\lim_{\varepsilon \rightarrow 0} \oint_{\partial D_\varepsilon(p_i)} K^{n+1} \bar{\omega} = (-2\pi \sqrt{-1}) \text{Res}_{p_i}(\omega) \mu^{n+1}.$$

Therefore we obtain

$$\begin{aligned} \mathcal{C}_n &= \frac{2\sqrt{-1}}{n+1} [(2\pi \sqrt{-1}) \sum_{i=1}^I \text{Res}_{p_i}(\omega) \mu^{n+1} + \\ &\quad (2\pi \sqrt{-1}) \sum_{s=1}^S \text{Res}_{e_s}(\omega) (-2\mu)^{n+1} + (2\pi \sqrt{-1}) \sum_{j=L+1}^J \text{Res}_{q_j}(\omega) (-2\mu)^{n+1}] \\ &= \frac{(-4\pi)}{n+1} \mu^{n+1} [\sum_{i=1}^I \text{Res}_{p_i}(\omega) + \sum_{s=1}^S \text{Res}_{e_s}(\omega) (-2)^{n+1} + \sum_{j=L+1}^J \text{Res}_{q_j}(\omega) (-2)^{n+1}]. \end{aligned}$$

Since

$$\sum_{i=1}^I \text{Res}_{p_i}(\omega) + \sum_{s=1}^S \text{Res}_{e_s}(\omega) + \sum_{j=L+1}^J \text{Res}_{q_j}(\omega) = 0$$

and

$$\begin{aligned} \text{Res}_{e_s}(\omega) &= -\frac{1}{3\mu^2}, \quad s = 1, 2, \dots, S, \\ \text{Res}_{q_j}(\omega) &= -\frac{\alpha_j}{3\mu^2}, \quad j = L+1, L+2, \dots, J, \end{aligned}$$

$$\mathcal{C}_n = \frac{2}{3(n+1)} \mu^{n-1} [(-2)^{n+1} - 1] \alpha_{max}, \quad (78)$$

where $\alpha_{max} = 2\pi(S + \sum_{j=L+1}^J \alpha_j)$ means the sum of the angles of the maximum points of K . By (78), $\mathcal{C}_n > 0$, $n = 0, 1, 2, \dots$, and

$$\begin{aligned} \mathcal{C}_0 &= \text{Area}(g) = -\frac{2}{\mu} \alpha_{max}, \\ \mathcal{C}_1 &= \int_{\Sigma'} K dg = \alpha_{max} = 2\pi(S + \sum_{j=L+1}^J \alpha_j), \\ \mathcal{C}_2 &= \mathcal{C}(g) = -2\mu \alpha_{max}. \end{aligned}$$

3.2 Proof of the sufficiency of the main theorem

In this section, we will follow the steps in the proof of the sufficiency of the main theorem in [7]. Since $\omega + \bar{\omega}$ is exact on $\Sigma' = \Sigma \setminus \{p_1, p_2, \dots, p_I, q_1, q_2, \dots, q_J, e_1, e_2, \dots, e_S\}$, we suppose that $\omega + \bar{\omega} = df_0$, where f_0 is a smooth function on Σ' . And we let $\mu = -\frac{1}{\sqrt{-3\Lambda}}$.

Step 1. Consider the equation on Σ'

$$\begin{cases} \frac{(-3)dK}{(K-\mu)^2(K+2\mu)} = \omega + \bar{\omega} \\ K(p_0) = K_0, \quad p_0 \in \Sigma', \quad \mu < K_0 < -2\mu. \end{cases} \quad (79)$$

Claim 1: (79) has a unique real smooth solution K on Σ' such that $\mu < K < -2\mu$.

Proof. First

$$\frac{1}{(K-\mu)^2(K+2\mu)} = \frac{1}{9\mu^2} \left[\frac{1}{K+2\mu} - \frac{1}{K-\mu} + \frac{3\mu}{(K-\mu)^2} \right].$$

Since $\omega + \bar{\omega} = df_0$, (79) is equivalent to

$$\begin{cases} \left[\frac{1}{K+2\mu} - \frac{1}{K-\mu} + \frac{3\mu}{(K-\mu)^2} \right] dK = d\frac{f_0}{\Lambda} \\ K(p_0) = K_0. \end{cases} \quad (80)$$

Also let

$$\sigma(t) = \ln(-2\mu - t) - \ln(t - \mu) - \frac{3\mu}{t - \mu}, \quad t \in (\mu, -2\mu).$$

Then by the same argument in the proof of Proposition 3.4, we can define a real function K^* on Σ' such that $\mu < K^* < -2\mu$ and

$$\sigma(K^*) = \frac{f_0}{\Lambda} + A_0, \quad (81)$$

where $A_0 = \sigma(K_0) - \frac{f_0(p_0)}{\Lambda}$. By the implicit theorem, $K^* \in C^\infty(\Sigma')$. Since K^* satisfies (81),

$$d\sigma(K^*) = d\frac{f_0}{\Lambda}$$

and $\sigma(K^*(p_0)) = \sigma(K_0)$, which means $K^*(p_0) = K_0$. Therefore K^* is a solution of (80). By the uniqueness of the solutions of (81), K^* is the uniqueness solution of (80). We prove the claim. \square

In the following, we use K to denote the solution of (79). Since each q_l , $l = 1, 2, \dots, L$, is a zero of ω , f_0 can be smoothly extended to q_l and K can also be smoothly extended to q_l with $\mu < K(q_l) < -2\mu$. Next we have the following claim:

Claim 2: K can be continuously extended to p_i , $i = 1, 2, \dots, I$, with $K(p_i) = \mu$, to $q_{l'}$, $l' = L+1, L+2, \dots, J$, with $K(q_{l'}) = -2\mu$ and to e_s , $s = 1, 2, \dots, S$, with $K(e_s) = -2\mu$.

Proof. First pick any p_i and let (U, z) be a local complex coordinate chart around p_i such that $U \setminus \{p_i\} \subset \Sigma'$, $z(U)$ is a disk D and $z(p_i) = 0$. Then suppose

$$\omega|_{U \setminus \{p_i\}} = \frac{\lambda_{-1}}{z} dz + d\eta_1 = \frac{\eta_2(z)}{z} dz, \quad (82)$$

where λ_{-1} is the residue of ω at p_i , η_1 is a holomorphic function on D and $\eta_2(z)$ is also a holomorphic function on D with $\eta_2(0) = \lambda_{-1}$. Then

$$(\omega + \bar{\omega})|_{U \setminus \{p_i\}} = \lambda_{-1} d \ln |z|^2 + 2d \operatorname{Re}(\eta_1) = df_0.$$

Therefore

$$f_0 = \lambda_{-1} \ln |z|^2 + 2 \operatorname{Re}(\eta_1) + a^* \quad \text{on } D \setminus \{0\},$$

where a^* is a real constant, or equivalently,

$$\frac{f_0}{\Lambda} = \frac{\lambda_{-1}}{\Lambda} \ln |z|^2 + 2 \operatorname{Re}\left(\frac{\eta_1}{\Lambda}\right) + \frac{a^*}{\Lambda} \quad \text{on } D \setminus \{0\}. \quad (83)$$

Substitute (83) into (81) to get on $D \setminus \{0\}$

$$\begin{aligned} \ln(-2\mu - K) - \ln(K - \mu) - \frac{3\mu}{K - \mu} = \\ \frac{\lambda_{-1}}{\Lambda} \ln |z|^2 + 2 \operatorname{Re}\left(\frac{\eta_1}{\Lambda}\right) + \frac{a^*}{\Lambda} + A_0, \end{aligned}$$

where $A_0 = \sigma(K_0) - \frac{f_0(p_0)}{\Lambda}$, or equivalently, on $D \setminus \{0\}$

$$(-2\mu - K) \frac{1}{K - \mu} e^{-\frac{3\mu}{K - \mu}} = A^* |z|^{\frac{2\lambda_{-1}}{\Lambda}} e^{2 \operatorname{Re}(\frac{\eta_1}{\Lambda})}, \quad (84)$$

where A^* is a positive constant. Suppose that there is a sequence $\{z_n\}$ in $D \setminus \{0\}$ such that $z_n \rightarrow 0$ as $n \rightarrow \infty$ and $\lim_{n \rightarrow \infty} K(z_n) = b^*$. Then $\mu \leq b^* \leq -2\mu$. If $b^* \neq \mu$, then substitute $\{z_n\}$

into (84), let $n \rightarrow \infty$ and take limits. The limit of the left side is $(-2\mu - b^*) \frac{1}{b^* - \mu} e^{-\frac{3\mu}{b^* - \mu}}$, the limit of the right side is $+\infty$ (note $\lambda_{-1} > 0$). It is a contradiction. Hence $b^* = \mu$, which shows

$$\lim_{z \rightarrow 0} K = \mu.$$

Similarly, we can prove that

$$\lim_{p \rightarrow q_{l'}} K(p) = -2\mu \quad \text{and} \quad \lim_{p \rightarrow e_s} K(p) = -2\mu.$$

Then we prove the claim. □

Step 2. Define a metric on Σ'

$$g = -\frac{4}{3}(K - \mu)^2(K + 2\mu)\omega\bar{\omega}. \quad (85)$$

Claim 3. g is an HCMU metric on Σ' and K is just the Gauss curvature of g .

Proof. Let (U, z) be a local complex coordinate chart on Σ' . Suppose

$$\omega = \rho(z)dz \quad \text{on } U.$$

Then

$$g|_U = -\frac{4}{3}(K - \mu)^2(K + 2\mu)|\rho|^2|dz|^2.$$

Let

$$e^{2\varphi} = -\frac{4}{3}(K - \mu)^2(K + 2\mu)|\rho|^2,$$

then

$$\varphi = \frac{1}{2} \ln \frac{4(K - \mu)^2(-2\mu - K)|\rho|^2}{3}.$$

Therefore

$$\varphi_z = \frac{3\rho(K + \mu)K_z + (K + 2\mu)(K - \mu)\rho_z}{2(K - \mu)(K + 2\mu)\rho}. \quad (86)$$

By (79),

$$K_z = -\frac{1}{3}(K - \mu)^2(K + 2\mu)\rho. \quad (87)$$

Substitute (87) into (86) to get

$$\varphi_z = \frac{1}{2}[-\rho(K^2 - \mu^2) + \frac{\rho_z}{\rho}].$$

Then

$$\varphi_{z\bar{z}} = -\rho K K_{\bar{z}} = \frac{1}{3}K(K - \mu)^2(K + 2\mu)|\rho|^2.$$

Therefore

$$-\Delta\varphi = Ke^{2\varphi},$$

which shows K is just the Gauss curvature of g . Meanwhile,

$$K_{\bar{z}}\rho = -\frac{1}{3}(K - \mu)^2(K + 2\mu)|\rho|^2 = \frac{1}{4}e^{2\varphi},$$

so

$$e^{-2\varphi}K_{\bar{z}} = \frac{1}{4\rho},$$

which means ∇K is a holomorphic vector field on Σ' . Hence g is an HCMU metric on Σ' . \square

Step 3. Claim 4. g is smooth at e_s , $s = 1, 2, \dots, S$, and satisfies the angle condition at p_i , $i = 1, 2, \dots, I$, and q_j , $j = 1, 2, \dots, J$.

Proof. Pick any e_s and let (U, z) be a local complex coordinate chart around e_s such that $U \setminus \{e_s\} \subset \Sigma'$, $z(U)$ is a disk D and $z(e_s) = 0$. Suppose that

$$\omega|_{U \setminus \{e_s\}} = \frac{\Lambda}{z}dz + d\eta_1 = \frac{\eta_2(z)}{z}dz,$$

where η_1 is a holomorphic function on D and $\eta_2(z)$ is a nonvanishing holomorphic function on D . Then

$$g|_{U \setminus \{e_s\}} = e^{2\varphi}|dz|^2 = -\frac{4}{3}(K - \mu)^2(K + 2\mu)\frac{|\eta_2(z)|^2}{|z|^2}|dz|^2.$$

On the other hand, similar to (84), on $D \setminus \{0\}$

$$(-2\mu - K)\frac{1}{K - \mu}e^{-\frac{3\mu}{K - \mu}} = B_1|z|^2e^{2\operatorname{Re}(\frac{\eta_1}{\Lambda})},$$

where B_1 is a positive constant. Then

$$e^{2\varphi} = B_2(K - \mu)^3e^{[2\operatorname{Re}(\frac{\eta_1}{\Lambda}) + \frac{3\mu}{K - \mu}]}|\eta_2(z)|^2,$$

where $B_2 = \frac{4}{3}B_1$. Therefore $e^{2\varphi}$ is continuous and positive on D . Next consider the system of equations on $D \setminus \{0\}$:

$$\begin{cases} K_{\bar{z}} = \frac{z}{4\eta_2(z)}e^{2\varphi} \\ e^{2\varphi} = B_2(K - \mu)^3 e^{[2\operatorname{Re}(\frac{\eta_1}{\Lambda}) + \frac{3\mu}{K-\mu}]} |\eta_2(z)|^2. \end{cases} \quad (88)$$

Apply Lemma 3.8 to the first equation of (88) to get $K \in C^1(D)$. Then by the bootstrap technique, $K \in C^\infty(D)$ and $e^{2\varphi} \in C^\infty(D)$. And

$$-\Delta\varphi = Ke^{2\varphi}, \quad K_{\bar{z}} = \frac{z}{4\eta_2(z)}e^{2\varphi}$$

hold on D , which shows g is actually an HCMU metric on $\Sigma^* = \Sigma \setminus \{p_1, p_2, \dots, p_I, q_1, q_2, \dots, q_J\}$.

Pick any p_i and let (V, w) be a local complex coordinate chart around p_i such that $V \setminus \{p_i\} \subset \Sigma'$, $w(V)$ is a disk \hat{D} and $w(p_i) = 0$. Suppose

$$\omega|_{V \setminus \{p_i\}} = \frac{\hat{\lambda}_{-1}}{w}dw + d\hat{\eta}_1 = \frac{\hat{\eta}_2(w)}{w}dw,$$

where $\hat{\lambda}_{-1} = \operatorname{Res}_{p_i}(\omega)$, $\hat{\eta}_1$ is a holomorphic function on \hat{D} , $\hat{\eta}_2(w)$ is also a holomorphic function on \hat{D} with $\hat{\eta}_2(0) = \hat{\lambda}_{-1}$. Then on $\hat{D} \setminus \{0\}$

$$\ln(-2\mu - K) - \ln(K - \mu) - \frac{3\mu}{K - \mu} = \frac{2\hat{\lambda}_{-1}}{\Lambda} \ln|w| + 2\operatorname{Re}\left(\frac{\hat{\eta}_1}{\Lambda}\right) + \hat{a},$$

where \hat{a} is a real constant. Therefore

$$\lim_{w \rightarrow 0} (K - \mu) \ln|w| = \hat{A}, \quad (89)$$

where \hat{A} is a nonzero real number. On the other hand,

$$g|_{V \setminus \{p_i\}} = e^{2\hat{\varphi}}|dw|^2 = -\frac{4}{3}(K - \mu)^2(K + 2\mu) \frac{|\hat{\eta}_2(w)|^2}{|w|^2} |dw|^2,$$

that is,

$$e^{2\hat{\varphi}} = -\frac{4}{3}(K - \mu)^2(K + 2\mu) \frac{|\hat{\eta}_2(w)|^2}{|w|^2}.$$

Then

$$\begin{aligned} \hat{\varphi} &= \frac{1}{2} \ln \left[-\frac{4}{3}(K - \mu)^2(K + 2\mu) \frac{|\hat{\eta}_2(w)|^2}{|w|^2} \right] \\ &= \ln(K - \mu) - \ln|w| + \frac{1}{2} \ln \frac{4(-2\mu - K)|\hat{\eta}_2(w)|^2}{3}. \end{aligned}$$

By (89),

$$\lim_{w \rightarrow 0} \frac{\hat{\varphi} + \ln|w|}{\ln|w|} = 0,$$

which shows p_i is a cusp singularity of g .

Next pick any q_l and let (W, ξ) be a local complex coordinate chart around q_l such that $W \setminus \{q_l\} \subset \Sigma'$, $\xi(W)$ is a disk D' and $\xi(q_l) = 0$. Since q_l is a zero of ω with $\operatorname{ord}_{q_l}(\omega) = \alpha_l - 1$, suppose

$$\omega|_W = \xi^{\alpha_l - 1} H(\xi) d\xi,$$

where $H(\xi)$ is a nonvanishing holomorphic function on D' . Therefore

$$g|_{W \setminus \{q_l\}} = -\frac{4}{3}(K - \mu)^2(K + 2\mu)|\xi|^{2\alpha_l - 2}|H(\xi)|^2|d\xi|^2.$$

Since $\mu < K(q_l) < -2\mu$, g has a conical singularity at q_l with the singular angle $2\pi\alpha_l$.

Finally pick any $q_{l'}$ and let (Y, ζ) be a local complex coordinate chart around $q_{l'}$ such that $Y \setminus \{q_{l'}\} \subset \Sigma'$, $\zeta(Y)$ is a disk \tilde{D} and $\zeta(q_{l'}) = 0$. Suppose that

$$\omega|_{Y \setminus \{q_{l'}\}} = \frac{\Lambda\alpha_{l'}}{\zeta}d\zeta + d\tilde{\eta}_1 = \frac{\tilde{\eta}_2(\zeta)}{\zeta}d\zeta,$$

where $\tilde{\eta}_1$ is a holomorphic function on \tilde{D} and $\tilde{\eta}_2(\zeta)$ is also a holomorphic function on \tilde{D} with $\tilde{\eta}_2(0) = \Lambda\alpha_{l'}$. Then on $\tilde{D} \setminus \{0\}$

$$\ln(-2\mu - K) - \ln(K - \mu) - \frac{3\mu}{K - \mu} = \alpha_{l'} \ln|\zeta|^2 + 2Re(\frac{\tilde{\eta}_1}{\Lambda}) + \tilde{a},$$

where \tilde{a} is a constant. On the other hand,

$$g|_{Y \setminus \{q_{l'}\}} = -\frac{4}{3}(K - \mu)^2(K + 2\mu)\frac{|\tilde{\eta}_2(\zeta)|^2}{|\zeta|^2}|d\zeta|^2. \quad (90)$$

Then substitute

$$-2\mu - K = \tilde{A}|\zeta|^{2\alpha_{l'}}(K - \mu)e^{[2Re(\frac{\tilde{\eta}_1}{\Lambda}) + \frac{3\mu}{K - \mu}]}$$

into (90) to get

$$g|_{Y \setminus \{q_{l'}\}} = A^*(K - \mu)^3 e^{[2Re(\frac{\tilde{\eta}_1}{\Lambda}) + \frac{3\mu}{K - \mu}]} |\tilde{\eta}_2(\zeta)|^2 |\zeta|^{2\alpha_{l'} - 2} |d\zeta|^2,$$

where \tilde{A} and A^* are constants. Therefore g has a conical singularity at $q_{l'}$ with the singular angle $2\pi\alpha_{l'}$. We prove the claim. \square

Step 4. g has finite area and finite Calabi energy.

We can use the same method in calculating \mathcal{C}_n to get g has finite area and finite Calabi energy.

Hence we finish the proof of the sufficiency of Theorem 1.2.

4 Existence of a kind of meromorphic 1-forms on a Riemann surface

We see that the character 1-form ω of an HCMU metric on a compact Riemann surface which has cusp singularities and conical singularities must have the following properties:

1. ω only has simple poles,
2. The residue of ω is a real number at each pole,
3. $\omega + \bar{\omega}$ is exact on $\Sigma \setminus \{\text{poles of } \omega\}$.

In fact, on any Riemann surface(compact or noncompact), this kind of meromorphic 1-form exists.

Theorem 4.1 ([10]). *Let Σ be a Riemann surface and p, q be two distinct points on Σ . Then there exists a meromorphic 1-form ω on Σ such that*

1. ω only has two simple poles at p and q with $\text{Res}_p(\omega) = 1$ and $\text{Res}_q(\omega) = -1$;
2. $\omega + \bar{\omega}$ is exact on $\Sigma \setminus \{p, q\}$.

By Theorem 4.1, we can get the following theorem:

Theorem 4.2. *Let Σ be a Riemann surface, p_1, p_2, \dots, p_n be $n(n \geq 2)$ points on Σ and $\lambda_1, \lambda_2, \dots, \lambda_n$ be n nonzero real numbers with $\sum_{i=1}^n \lambda_i = 0$. Then there exists a meromorphic 1-form ω on Σ such that*

1. ω only has simple poles at p_1, p_2, \dots, p_n with $\text{Res}_{p_i}(\omega) = \lambda_i, i = 1, 2, \dots, n$;
2. $\omega + \bar{\omega}$ is exact on $\Sigma \setminus \{p_1, p_2, \dots, p_n\}$.

Now let Σ be a compact Riemann surface. ω is a given meromorphic 1-form which satisfies the conditions in Theorem 4.2. Then following the proof of the sufficiency of Theorem 1.2, we can get there exists an HCMU metric which has cusp singularities and conical singularities and just has ω as the character 1-form. Meanwhile we can see it is possible that different HCMU metrics (even HCMU metrics only with conical singularities and HCMU metrics with cusp singularities and conical singularities) have the same character 1-form.

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